

THE CUBIC FOURTH-ORDER SCHRÖDINGER EQUATION

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ABSTRACT. Fourth-order Schrödinger equations have been introduced by Karpman and Shagalov to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. In this paper we investigate the cubic defocusing fourth order Schrödinger equation

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0$$

in arbitrary space dimension \mathbb{R}^n for arbitrary initial data. We prove that the equation is globally well-posed when $n \leq 8$ and ill-posed when $n \geq 9$, with the additional important information that scattering holds true when $5 \leq n \leq 8$.

1. INTRODUCTION

Fourth-order Schrödinger equations have been introduced by Karpman [15] and Karpman and Shagalov [16] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations have been studied from the mathematical viewpoint in Fibich, Ilan and Papanicolaou [8] who describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Related references are by Ben-Artzi, Koch, and Saut [3] who gave sharp dispersive estimates for the biharmonic Schrödinger operator, Guo and Wang [11] who proved global well-posedness and scattering in H^s for small data, Hao, Hsiao and Wang [12, 13] who discussed the Cauchy problem in a high-regularity setting, and Segata [35] who proved scattering in the case the space dimension is one. We refer also to Pausader [28, 29] where the energy critical case for radially symmetrical initial data is discussed. The defocusing case like in (1.1) below is discussed in Pausader [28] for radially symmetrical initial data. The focusing case, following the beautiful results of Kenig and Merle [18, 19], is settled in Pausader [29] still for radially symmetrical initial data.

We focus in this paper on the study of the initial value problem for the cubic fourth-order defocusing equation in arbitrary space dimension \mathbb{R}^n , $n \geq 1$, without assuming radial symmetry for the initial data. The equation is written as

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0, \quad (1.1)$$

where $u = I \times \mathbb{R}^n \rightarrow \mathbb{C}$ is a complex valued function, and $u|_{t=0} = u_0$ is in H^2 , the space of L^2 functions whose first and second derivatives are in L^2 . The equation is critical when $n = 8$ because of the criticality of the Sobolev embedding $H^2 \subset L^4$ in this dimension, and it enjoys rescaling invariance leaving the energy and \dot{H}^2 -norm unchanged. Let \mathcal{S} be the space of Schwartz functions. The theorem we prove in this paper provides a complete picture of global well-posedness for (1.1). It is stated as follows.

Theorem 1.1. *Assume $1 \leq n \leq 8$. Then for any $u_0 \in H^2$ there exists a global solution $u \in C(\mathbb{R}, H^2)$ of (1.1) with initial data $u(0) = u_0$. Moreover, for any $t \in \mathbb{R}$, the mapping $u(0) \mapsto u(t)$ is analytic from H^2 into itself. On the contrary, if $n \geq 9$ then the Cauchy problem for (1.1) is ill-posed in H^2 in the sense that for any $\varepsilon > 0$, there exist $u_0 \in \mathcal{S}$, $t_\varepsilon \in (0, \varepsilon)$, and $u \in C([0, \varepsilon], H^2)$ a solution of (1.1) with initial data u_0 such that $\|u_0\|_{H^2} < \varepsilon$ while $\|u(t_\varepsilon)\|_{H^2} > \varepsilon^{-1}$. Besides, if $5 \leq n \leq 8$, then scattering holds true in H^2 for (1.1) and the scattering operator is analytic.*

The fourth-order dispersion scaling property leads to the heuristic that smooth solutions of the free homogeneous equation have their L^∞ norm which decays like $t^{-\frac{n}{4}}$. However, the situation is not so transparent and all frequency parts of the function have their L^∞ -norm that decays much faster, like $t^{-\frac{n}{2}}$, but at a rate which depends on the frequency. Uniformly, the rate of decay $t^{-\frac{n}{4}}$ is the best possible, but it is not optimal when the solution is localized in frequency. As one will see, there are various differences between the dispersion behaviors of second-order Schrödinger equations and of (1.1).

Our paper is organised as follows. We fix notations in Section 2 and recall preliminary results from Pausader [28] in Section 3. In Section 4, we prove that the Cauchy problem is ill-posed when $n \geq 9$. In order to do so we use a low-dispersion regime argument which was essentially given in Christ, Colliander and Tao [6]. We also refer to Lebeau [24, 25], Alazard and Carles [1], Carles [4] and Thomann [38, 39] for other results in different settings. Starting from Section 5 we focus on the energy-critical case, and so on the $n = 8$ part of our theorem (the equation is subcritical when $n \leq 7$). We prove in Section 5, using important ideas of concentration compactness developed in Kenig and Merle [18] and Killip, Tao and Visan [23], that any failure of global wellposedness implies the existence of some special solutions satisfying three possible scenarii. The remaining part of the analysis consists in excluding these hypothetical special solutions working at the level of \dot{H}^2 -solutions. The first scenario is that there is a self-similar-like solution. It is not consistent with conservation of energy, conservation of local mass and compactness up to rescaling. We exclude this scenario in Section 6. The two other scenarii are that there is a soliton-like solution or that there is a low-to-high cascade-like solution. In these two scenarii the solution is away from the L^2 -like region, namely we have that $h \leq 1$ with respect to the notation of Theorem 5.1. We use this to prove an interaction Morawetz estimate in Sections 7 and 8, following previous analysis from Colliander, Keel, Staffilani, Takaoka and Tao [7], Ryckman and Visan [32] and Visan [40]. The estimate we prove is not an a priori estimate. A major difficulty is that the estimate scales like the $\dot{H}^{\frac{1}{4}}$ -norm and thus creates a $7/4$ -difference in scaling with the \dot{H}^2 -norm control we have. In Section 9, we exclude soliton-like solution by proving that it is not consistent with the frequency-localized interaction Morawetz estimates and compactness up to rescaling. The last scenario is excluded in Section 10 by proving that any low-to-high-like solution has an unexpected L^2 -regularity. Then, conservation of L^2 -norm, frequency-localized interaction Morawetz estimates and conservation of energy allows us to exclude this existence of low-to-high cascade-like cascade solutions. Finally, in Section 11, we prove the scattering part of Theorem 1.1.

As a remark, with the arguments we develop here and adaptations of the analysis in Visan [40], global well-posedness and scattering in Theorem 1.1 continue to hold

true when $n \geq 8$ and the cubic nonlinearity is replaced by the n -dimensional energy-critical nonlinearity with total power $(n+4)/(n-4)$. We also refer to Miao, Xu and Zhao [27] for another proof in high dimensions $n \geq 9$ following previous work by Killip and Visan [22]. For radially symmetrical data, see Pausader [28], this is also true in any dimension $n \geq 5$.

2. NOTATIONS

We fix notations we use throughout the paper. In what follows, we write $A \lesssim B$ to signify that there exists a constant C depending only on n such that $A \leq CB$. When the constant C depends on other parameters, we indicate this by a subscript, for example, $A \lesssim_u B$ means that the constant may depend on u . Similar notations hold for \gtrsim . Similarly we write $A \simeq B$ when $A \lesssim B \lesssim A$.

We let $L^q = L^q(\mathbb{R}^n)$ be the usual Lebesgue spaces, and $L^r(I, L^q)$ be the space of measurable functions from an interval $I \subset \mathbb{R}$ to L^q whose $L^r(I, L^q)$ norm is finite, where

$$\|u\|_{L^r(I, L^q)} = \left(\int_I \|u(t)\|_{L^q}^r dt \right)^{\frac{1}{r}}.$$

When there is no risk of confusion we may write $L^q L^r$ instead of $L^q(I, L^r)$. Two important conserved quantities of equation (1.1) are the mass and the energy. The mass is defined by

$$M(u) = \int_{\mathbb{R}^n} |u(x)|^2 dx \quad (2.1)$$

and the energy is defined by

$$E(u) = \int_{\mathbb{R}^n} \left(\frac{|\Delta u(x)|^2}{2} + \frac{|u(x)|^4}{4} \right) dx. \quad (2.2)$$

In what follows we let $\mathcal{F}f = \hat{f}$ be the Fourier transform of f given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{i\langle y, \xi \rangle} dy$$

for all $\xi \in \mathbb{R}^n$. The biharmonic Schrödinger semigroup is defined for any tempered distribution g by

$$e^{it\Delta^2} g = \mathcal{F}^{-1} e^{it|\xi|^4} \mathcal{F}g. \quad (2.3)$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be supported in the ball $B(0, 2)$, and such that $\psi = 1$ in $B(0, 1)$. For any dyadic number $N = 2^k, k \in \mathbb{Z}$, we define the following Littlewood-Paley operators:

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &= \psi(\xi/N) \hat{f}(\xi), \\ \widehat{P_{> N} f}(\xi) &= (1 - \psi(\xi/N)) \hat{f}(\xi), \\ \widehat{P_N f}(\xi) &= (\psi(\xi/N) - \psi(2\xi/N)) \hat{f}(\xi). \end{aligned} \quad (2.4)$$

Similarly we define $P_{< N}$ and $P_{\geq N}$ by the equations

$$P_{< N} = P_{\leq N} - P_N \quad \text{and} \quad P_{\geq N} = P_{> N} + P_N$$

These operators commute one with another. They also commute with derivative operators and with the semigroup $e^{it\Delta^2}$. In addition they are self-adjoint and

bounded on L^p for all $1 \leq p \leq \infty$. Moreover, they enjoy the following Bernstein property:

$$\begin{aligned} \|P_{\geq N} f\|_{L^p} &\lesssim_s N^{-s} \| |\nabla|^s P_{\geq N} f \|_{L^p} \lesssim_s N^{-s} \| |\nabla|^s f \|_{L^p} \\ \| |\nabla|^s P_{\leq N} f \|_{L^p} &\lesssim_s N^s \| P_{\leq N} f \|_{L^p} \lesssim_s N^s \| f \|_{L^p} \\ \| |\nabla|^{\pm s} P_N f \|_{L^p} &\lesssim_s N^{\pm s} \| P_N f \|_{L^p} \lesssim_s N^{\pm s} \| f \|_{L^p} \end{aligned} \quad (2.5)$$

for all $s \geq 0$, and all $1 \leq p \leq \infty$, independently of f , N , and p , where $|\nabla|^s$ is the classical fractional differentiation operator. We refer to Tao [36] for more details. Given $a \geq 1$, we let a' be the conjugate of a , so that $\frac{1}{a} + \frac{1}{a'} = 1$.

Several norms have to be considered in the analysis of the critical case of (1.1). For $I \subset \mathbb{R}$ an interval, they are defined as

$$\begin{aligned} \|u\|_{M(I)} &= \|\Delta u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2n(n+4)}{n^2+16}})}, \\ \|u\|_{W(I)} &= \|\nabla u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2n(n+4)}{n^2-2n+8}})}, \\ \|u\|_{Z(I)} &= \|u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2(n+4)}{n-4}})}, \text{ and} \\ \|u\|_{N(I)} &= \|\nabla u\|_{L^2(I, L^{\frac{2n}{n+2}})}. \end{aligned} \quad (2.6)$$

Accordingly, we let $M(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. We adopt similar definitions for W , Z , and N . We also need the following stronger norms in order to fully exploit the Strichartz estimates in Section 3. Following standard notations, we say that a pair (q, r) is Schrödinger-admissible, for short S -admissible, if $2 \leq q, r \leq \infty$, $(q, r, n) \neq (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \quad (2.7)$$

We define the full Strichartz norm of regularity s by

$$\|u\|_{\dot{S}^s(I)} = \sup_{(a,b)} \left(\sum_N N^{2s+\frac{4}{a}} \|P_N u\|_{L^a(I, L^b)}^2 \right)^{\frac{1}{2}}, \quad (2.8)$$

where the supremum is taken over all S -admissible pairs (a, b) as in (2.7), $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an interval. We also define the dual norm,

$$\|h\|_{\dot{S}^s(I)} = \inf_{(a,b)} \left(\sum_N N^{2s-\frac{4}{a}} \|P_N h\|_{L^{a'}(I, L^{b'})}^2 \right)^{\frac{1}{2}} \quad (2.9)$$

where again, the infimum is taken over all S -admissible pairs (a, b) as in (2.7), $s \in \mathbb{R}$, and I is an interval. We let $\dot{S}^s(I)$ be the set of tempered distributions of finite $\dot{S}^s(I)$ -norm. Finally, for a product $\pi = \Pi_i a_i$, we use the notation $\mathcal{O}(\pi)$ to denote an expression which is schematically like π , i.e. that is a finite combination of products $\pi' = \Pi_i b_i$ where in each π' , each b_i stands for a_i or for \bar{a}_i .

As a remark, if $n = 8$, then there is a rescaling invariance rule for (1.1) given by

$$u \mapsto \tau_{(h, t_0, x_0)} u = h^2 u(h^4(t - t_0), h(x - x_0)) \quad (2.10)$$

which sends a solution of (1.1) with initial data $u(0) = u_0$ to another solution with data at time $t = t_0$ given by

$$g_{(h,x_0)}u_0 = h^2u_0(h(x - x_0)), \quad (2.11)$$

and which leaves the energy and \dot{H}^2 -norm unchanged:

$$E(\tau_{(h,t_0,x_0)}u) = E(u) \quad \text{and} \quad \|g_{(h,x_0)}u_0\|_{\dot{H}^2} = \|u_0\|_{\dot{H}^2}$$

for all u_0, u, h, t_0, x_0 . The associated loss of compactness makes that (1.1) is particularly difficult to handle in the critical dimension $n = 8$. In the radially symmetrical case the difficulty was overcome in Pausader [28]. We prove here that we can get rid of the radially symmetrical assumption.

3. PRELIMINARY RESULTS

We recall results from Pausader [28]. We refer to Pausader [28] for their proof. A first result from Pausader [28] is that the following fundamental Strichartz-type estimates hold true. Note that these estimates, because of the gain of derivatives, contradict the Galilean invariance one could have expected for the fourth order Schrödinger equation.

Proposition 3.1. *Let $u \in C(I, H^{-4})$ be a solution of*

$$i\partial_t u + \Delta^2 u + h = 0, \quad (3.1)$$

and $u(0) = u_0$. Then, for any S -admissible pairs (q, r) and (a, b) as in (2.7), and any $s \in \mathbb{R}$,

$$\| |\nabla|^s u \|_{L^q(I, L^r)} \lesssim \left(\| |\nabla|^{s-\frac{2}{q}} u_0 \|_{L^2} + \| |\nabla|^{s-\frac{2}{q}-\frac{2}{a}} h \|_{L^{a'}(I, L^{b'})} \right) \quad (3.2)$$

whenever the right hand side in (3.2) is finite.

A consequence of the Strichartz estimates (3.2) and of the commutation properties of the linear propagator $e^{it\Delta^2}$ is the following estimate, for any solution u as above:

$$\begin{aligned} \|u\|_{\dot{S}^s(I)} &\lesssim \|u_0\|_{\dot{H}^s} + \|h\|_{\dot{S}^s(I)} \\ &\lesssim \|u_0\|_{\dot{H}^s} + \| |\nabla|^{s-\frac{2}{a}} h \|_{L^{a'}(I, L^{b'})}, \end{aligned} \quad (3.3)$$

where (a, b) is an S -admissible pair as in (2.7), and the norms are defined in (2.8) and (2.9) above. A preliminary version of (3.2) was obtained in Kenig, Ponce and Vega [20]. Let $u \in C(I, \dot{H}^2)$ be defined on some interval I such that $0 \in I$ and such that $u \in L_{loc}^3(I \times \mathbb{R}^n)$. We say that u is a solution of (1.1) provided that the following equality holds in the sense of tempered distributions for all times:

$$u(t) = e^{it\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} (|u|^2 u)(s) ds. \quad (3.4)$$

Note that, by Strichartz estimates, if $u_0 \in L^2$ and $|u|^2 u \in L_{loc}^1(I, L^2)$, then (3.4) is equivalent to the fact that u solves (1.1) in H^{-4} with $u(0) = u_0$.

The following Propositions 3.2 and 3.3, still from Pausader [28], are important for the energy-critical case $n = 8$. Proposition 3.2 settles the question of local well-posedness. Proposition 3.3 settles the question of stability.

Proposition 3.2. *Let $n = 8$. There exists $\delta > 0$ such that for any initial data $u_0 \in \dot{H}^2$, and any interval $I = [0, T]$, if*

$$\|e^{it\Delta^2} u_0\|_{W(I)} < \delta, \quad (3.5)$$

then there exists a unique solution $u \in C(I, \dot{H}^2)$ of (1.1) with initial data u_0 . This solution has conserved energy, and satisfies $u \in \dot{S}^2(I)$. Moreover,

$$\|u\|_{\dot{S}^2(I)} \lesssim \|u_0\|_{\dot{H}^2} + \delta^3, \quad (3.6)$$

and if $u_0 \in H^2$, then $u \in \dot{S}^0(I) \cap \dot{S}^2(I)$,

$$\|u\|_{\dot{S}^0(I)} \lesssim \|u_0\|_{L^2},$$

and u has conserved mass. Besides, in this case, the solution depends continuously on the initial data in the sense that there exists δ_0 , depending on δ , such that, for any $\delta_1 \in (0, \delta_0)$, if $\|v_0 - u_0\|_{H^2} \leq \delta_1$, and if we let v be the local solution of (1.1) with initial data v_0 , then v is defined on I and $\|u - v\|_{\dot{S}^0(I)} \lesssim \delta_1$.

In addition to Proposition 3.2 we also have Proposition 3.3.

Proposition 3.3. *Let $n = 8$, $I \subset \mathbb{R}$ be a compact time interval such that $0 \in I$, and \tilde{u} be an approximate solution of (1.1) in the sense that*

$$i\partial_t \tilde{u} + \Delta^2 \tilde{u} + |\tilde{u}|^2 \tilde{u} = e \quad (3.7)$$

for some $e \in N(I)$. Assume that $\|\tilde{u}\|_{Z(I)} < +\infty$ and $\|\tilde{u}\|_{L^\infty(I, \dot{H}^2)} < +\infty$. There exists $\delta_0 > 0$, $\delta_0 = \delta_0(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I, \dot{H}^2)})$, such that if $\|e\|_{N(I)} \leq \delta$, and $u_0 \in \dot{H}^2$ satisfies

$$\|\tilde{u}(0) - u_0\|_{\dot{H}^2} \leq \Lambda \quad \text{and} \quad \|e^{it\Delta^2} (\tilde{u}(0) - u_0)\|_{W(I)} \leq \delta \quad (3.8)$$

for some $\delta \in (0, \delta_0]$, then there exists $u \in C(I, \dot{H}^2)$ a solution of (1.1) such that $u(0) = u_0$. Moreover, u satisfies

$$\begin{aligned} \|u - \tilde{u}\|_{W(I)} &\leq C\delta, \\ \|u - \tilde{u}\|_{\dot{S}^2} &\leq C(\Lambda + \delta), \quad \text{and} \\ \|u\|_{\dot{S}^2} &\leq C, \end{aligned} \quad (3.9)$$

where $C = C(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I, \dot{H}^2)})$ is a nondecreasing function of its arguments.

In our analysis, we need to consider \dot{H}^2 -solutions. These solutions do not satisfy conservation of mass. However the next proposition shows that there is still something remaining from that conservation law for these solutions. Proposition 3.4 shows that the local mass of a solution of (1.1) varies slowly in time provided that the radius R is sufficiently large. We define the local mass $M(u, B(x_0, R))$ over the ball $B(x_0, R)$ of a function $u \in L^2_{loc}$ by

$$M(u, B(x_0, R)) = \int_{\mathbb{R}^n} |u(x)|^2 \psi^4((x - x_0)/R) dx, \quad (3.10)$$

where, ψ is as in (2.4). Proposition 3.4 from Pausader [28], states as follows.

Proposition 3.4. *Let $n \geq 5$, and $u \in C(I, \dot{H}^2)$ be a solution of (1.1). Then we have that*

$$|\partial_t M(u(t), B(x_0, R))| \lesssim \frac{E(u)^{\frac{3}{4}}}{R} M(u(t), B(x_0, R))^{\frac{1}{4}} \quad (3.11)$$

for all $t \in I$.

We refer to Pausader [28] for a proof of the above propositions.

4. ILL-POSEDNESS RESULTS

In this section we use a quantitative analysis of the small dispersion regime to prove ill-posedness results for the cubic equation when $n > 8$. The idea is that now the equation is supercritical with respect to the regularity-setting in which we work, namely H^2 . Hence one can always use rescaling arguments to make any “separation-mechanism” between two different solutions happen sooner and sooner while making the H^2 -norm smaller and smaller. It remains then to find two solutions whose distance goes to ∞ as time evolves. To achieve this, we follow the proof in Christ, Colliander and Tao [6] by considering the small dispersion regime. See also Lebeau [24, 25] for previous results, and Alazard and Carles [1], Carles [4] and Thomann [38, 39] for instability results in different contexts.

Before we prove our theorem, we need the following lemma concerning the small dispersion regime.

Lemma 4.1. *Let $k > n/2$. Then, for any $\phi \in \mathcal{S}$, there exists $c > 0$ such that for any $\nu \in (0, 1)$, there exists a unique solution $w^\nu \in C([-T, T], H^k)$ of the problem*

$$i\partial_t w + \nu^4 \Delta^2 w + |w|^2 w = 0 \quad (4.1)$$

with initial data $w^\nu(0) = \phi$, where $T = c|\log \nu|^c$. Besides, the solution satisfies $w^\nu \in C([-T, T], H^p)$ for any p , and

$$\|w^\nu - w^0\|_{L^\infty([-T, T], H^k)} \lesssim_{\phi, k} \nu^3, \quad (4.2)$$

where

$$w^0(t, x) = \phi(x) \exp(i|\phi(x)|^2 t) \quad (4.3)$$

is a solution of the ODE formally obtained by setting $\nu = 0$ in (4.1).

Proof. Letting $u = w^\nu - w^0$, we see that u solves the Cauchy problem

$$i\partial_t u + \nu^4 \Delta^2 u = \nu^4 \Delta^2 w^0 + |w^0|^2 w^0 - |w^0 + u|^2 (w^0 + u) \quad (4.4)$$

with $u(0) = 0$. Let $k > n/2$ be given. Since $w^0 \in C^\infty(\mathcal{S})$, standard developments ensure that there exists a unique solution $u \in C([-t, t], H^k)$ to (4.4), and that u can be continued as long as $\|u\|_{H^k}$ remains bounded. Besides, $u \in C([-t, t], H^p)$ for any $p \geq 0$ (in the sense that t does not depend on p). Consequently, it suffices to prove that there exists $c > 0$ such that for any $s < c|\log \nu|^c$, we have that $\|u(s)\|_{H^k} \leq \nu^3$. Now, taking derivatives ∂^α of equation (4.4), multiplying by $\partial^\alpha \bar{u}$, taking the imaginary part and integrating, for all α such that $|\alpha| \leq k$, we get that

$$\begin{aligned} & \partial_s \|u(s)\|_{H^k}^2 \\ & \lesssim \|u\|_{H^k} \left(\nu^4 \|\Delta^2 w^0(s)\|_{H^k} + \| |w^0 + u|^2 (w^0 + u) - |w^0|^2 w^0 \|_{H^k} \right). \end{aligned} \quad (4.5)$$

By (4.3) we see that, for $p \geq 0$,

$$\|w^0\|_{H^p} \lesssim_{\phi, p} t^p. \quad (4.6)$$

Independently, since H^k is an algebra, we get that

$$\begin{aligned} \| |w^0 + u|^2 (w^0 + u) - |w^0|^2 w^0 \|_{H^k} & \lesssim \sum_{j=0}^2 \|\mathcal{O}\left((w^0)^j u^{3-j}\right)\|_{H^k} \\ & \lesssim \|u\|_{H^k} (1 + \|w^0\|_{H^k} + \|u\|_{H^k})^2. \end{aligned} \quad (4.7)$$

Now, using (4.5)–(4.7), we see that, in the sense of distributions,

$$\partial_s \|u(s)\|_{H^k} \lesssim_{\phi,k} \nu^4 (1 + |s|^{k+4}) + \|u(s)\|_{H^k} (1 + |s|^k + \|u(s)\|_{H^k})^2. \quad (4.8)$$

An application of Gromwall's lemma gives the bound

$$\|u(s)\| \lesssim_{k,\phi} \nu^4 \exp(C(1 + |s|^C)) \quad (4.9)$$

for all s such that $\|u(s)\|_{H^k} \leq 1$. By (4.9) we see that $\|u(s)\|_{H^k} \leq 1$ holds for all times $|s| \leq c|\log \nu|^c$, $c > 0$ sufficiently small. This gives (4.2) and finishes the proof of Lemma 4.1. \square

Now, we are in position to prove the main theorem of this section which states that the flow map $u_0 \mapsto u(t)$, from H^2 into H^2 which maps the initial data to the associated solution fails to be continuous at 0. As a remark, note that (4.10) is false when $n \leq 8$ since the H^2 -norm controls the energy.

Theorem 4.1. *Let $n > 8$. Given $\varepsilon > 0$, there exists a solution $u \in C([0, \varepsilon], H^2)$ such that*

$$\|u(0)\|_{H^2} < \varepsilon \quad \text{and} \quad \|u(t_\varepsilon)\|_{H^2} > \varepsilon^{-1}, \quad (4.10)$$

for some $t_\varepsilon \in (0, \varepsilon)$. Besides, we can choose u such that $u(0) \in \mathcal{S}$ and $u \in C([0, \varepsilon], H^k)$ for any $k > 0$.

Proof of Theorem 4.1. For $\phi \in \mathcal{S}$ and $\nu \in (0, 1]$, we let w^ν be the solution of equation (4.1) with initial data $w^\nu(0) = \phi$. By Lemma 4.1, we see that for $|s| \leq c|\log \nu|^c$, (4.2) holds true for w^0 as in (4.3). Now, for $\lambda \in (0, \infty)$, we let

$$u^{(\nu, \lambda)}(t, x) = \lambda^2 w^\nu(\lambda^4 t, \lambda \nu x). \quad (4.11)$$

Then $u^{(\nu, \lambda)}$ solves (1.1) with initial data $u^{(\nu, \lambda)}(0, x) = \lambda^2 \phi(\lambda \nu x)$. A simple calculation gives

$$\begin{aligned} \|u^{(\nu, \lambda)}(0)\|_{H^2}^2 &= \frac{\lambda^4}{(2\pi)^n} (\lambda \nu)^{-2n} \int_{\mathbb{R}^n} |\hat{\phi}(\xi/(\lambda \nu))|^2 (1 + |\xi|^2)^2 d\xi \\ &\lesssim \lambda^4 (\lambda \nu)^{-n} \left(\int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2 |\lambda \nu \eta|^4 d\eta + \int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2 d\eta \right) \\ &\lesssim_\phi \lambda^4 (\lambda \nu)^{4-n}, \end{aligned} \quad (4.12)$$

provided that $\lambda \nu \geq 1$. Now, given $\varepsilon > 0$, and $\nu > 0$, we fix

$$\lambda = \lambda_{\nu, \varepsilon} = (\varepsilon^2 \nu^{n-4})^{-\frac{1}{n-8}} \quad (4.13)$$

such that $\lambda^4 (\lambda \nu)^{4-n} = \varepsilon^2$, and $\lambda \nu = (\varepsilon \nu^2)^{-\frac{2}{n-8}} > 1$. Independently, by (4.3), we see that

$$\|w^0(t)\|_{\dot{H}^2} \gtrsim_\phi t^2 + O(t),$$

and, consequently, using (4.2), we get that for $|s| \leq c|\log \nu|^c$ sufficiently large independently of ν , there holds that

$$\|w^\nu(s)\|_{\dot{H}^2} \gtrsim_\phi s^2. \quad (4.14)$$

Consequently, using (4.11), (4.13) and (4.14) we get that

$$\begin{aligned} \|u^{(\nu, \lambda)}(\lambda^{-4} t)\|_{H^2}^2 &\geq \|u^{(\nu, \lambda)}(\lambda^{-4} t)\|_{\dot{H}^2}^2 \\ &\geq \lambda^4 (\lambda \nu)^{4-n} \|w^\nu(t)\|_{\dot{H}^2}^2 \\ &\gtrsim_\phi \varepsilon^2 t^4 \end{aligned} \quad (4.15)$$

for t sufficiently large. Now, given ε , we let $\nu > 0$ be sufficiently small such that

$$\begin{aligned} \varepsilon^2 t_\nu^4 &> \varepsilon^{-2}, \text{ for } t_\nu = c|\log \nu|^c, \text{ and} \\ \varepsilon^{\frac{16-n}{n-8}} \nu^{\frac{4(n-4)}{n-8}} &< \varepsilon. \end{aligned} \quad (4.16)$$

We choose $\lambda = \lambda_{\nu, \varepsilon}$ as in (4.13). Using (4.16), we get that $t_\varepsilon = \lambda^{-4} t_\nu < \varepsilon$, and then (4.12) and (4.15) give (4.10). This finishes the proof. \square

5. REDUCTION TO THREE SCENARI

From now on we start with the analysis of the energy-critical case $n = 8$. In this section we prove that the analysis can be reduced to the study of some very special solutions. In order to do so, we borrow ideas from previous works developed in the context of Schrödinger and wave equations by Bahouri and Gerard [2], Kenig and Merle [18], Keraani [21], Killip, Tao and Visan [23], and Tao, Visan and Zhang [37]. We refer also to Pausader [30] for a similar result developed in the context of the L^2 -critical fourth-order Schrödinger equation. For any $E > 0$, we let

$$\Lambda(E) = \sup\{\|u\|_{Z(I)}^6 : E(u) \leq E\}, \quad (5.1)$$

where the supremum is taken over all maximal-lifespan solutions $u \in C(I, \dot{H}^2)$ of (1.1) satisfying $E(u) \leq E$. In light of Proposition 3.2 and of the Strichartz estimates (3.2), we know that there exists $\delta > 0$ such that, for any $E \leq \delta$, $\Lambda(E) \lesssim_\delta E < +\infty$. Besides, Λ is clearly an increasing function of E . Hence, we can define

$$E_{max} = \sup\{E > 0 : \Lambda(E) < \infty\}. \quad (5.2)$$

The goal in Sections 5–10 is to prove that $E_{max} = +\infty$. Theorem 5.1 below is a first step in this direction.

Theorem 5.1. *Suppose that $E_{max} < +\infty$. There exists $u \in C(I, \dot{H}^2)$ a maximal-lifespan solution of energy exactly E_{max} such that the $Z(I')$ -norm of u is infinite for $I' = (T_*, 0)$ and $I' = (0, T^*)$, where $I = (T_*, T^*)$. Besides, there exist two smooth functions $h : I \rightarrow \mathbb{R}_+^*$ and $x : I \rightarrow \mathbb{R}^n$ such that*

$$K = \{g_{(h(t), x(t))} u(t) : t \in I\} \quad (5.3)$$

is precompact in \dot{H}^2 , where the transformation $g(t) = g_{(h(t), x(t))}$ is as in (2.11). Furthermore, one can assume that one of the following three scenarii holds true: (soliton-like solution) there holds $I = \mathbb{R}$ and $h(t) = 1$ for all t ; (double low-to-high cascade) there holds $\liminf_{t \rightarrow \bar{T}} h(t) = 0$ for $\bar{T} = T_, T^*$, and $h(t) \leq 1$ for all t ; (self-similar solution) there holds $I = (0, +\infty)$ and $h(t) = t^{\frac{1}{4}}$ for all t .*

As a remark, since $E(u) = E_{max}$, the solution u in Theorem 5.1 is such that $u \neq 0$. Assuming Propositions 6.1, 9.1 and 10.1 which exclude the three scenarii in Theorem 5.1, the following corollary holds true.

Corollary 5.1. *For any $E > 0$, there exists $C = C(E)$ such that, for any $u_0 \in \dot{H}^2$ satisfying $E(u_0) \leq E$, if $u \in C(I, \dot{H}^2)$ is the maximal solution of (1.1) with initial data $u(0) = u_0$, then $I = \mathbb{R}$ and $\|u\|_{\dot{S}^2(\mathbb{R})} \leq C$.*

Proof of Corollary 5.1. First, using [28, Proposition 2.6.], we see that a bound on the Z -norm of u implies a bound on the \dot{S}^2 -norm of u . Hence if Corollary 5.1 is false, then $E_{max} < +\infty$. Applying Theorem 5.1, we find a maximal solution satisfying one of the three scenarii in Theorem 5.1. Then, using Propositions 6.1, 9.1 and 10.1, we get a contradiction. Hence $E_{max} = +\infty$. \square

Now we prove Theorem 5.1.

Proof of Theorem 5.1. In several ways the proof is similar to the one developed in the L^2 -critical case in Pausader [30]. We prove the more general statement that Theorem 5.1 holds true in any dimension $n \geq 5$ when (1.1) is replaced by the \dot{H}^2 -critical equation. In particular, this is the case when $n = 8$. Therefore, in this proof, (1.1) always refers to the energy-critical equation in dimension n , and the energy E and Λ must be replaced by

$$E(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\Delta u(x)|^2 + \frac{n-4}{2n} |u(x)|^{\frac{2n}{n-4}} \right) dx \quad \text{and}$$

$$\Lambda(E) = \sup \{ \|u\|_{\dot{H}^2}^{\frac{2(n+4)}{n-4}} : E(u) \leq E \},$$

where the supremum is taken over all maximal solutions of the energy-critical equation of energy less or equal to E . Besides, the definition of τ and g as in (2.10) and (2.11) and Propositions 3.2 and 3.3 refer to their n -dimensional energy-critical counterparts. A consequence of the precised Sobolev's inequality in Gerard, Meyer and Oru [10] and of the Strichartz estimates (3.2) is that, for any $u_0 \in \dot{H}^2$,

$$\begin{aligned} \|e^{it\Delta^2} u_0\|_{Z(\mathbb{R})} &\lesssim \|e^{it\Delta^2} |\nabla| u_0\|_{L^{\frac{n-4}{n-2}} L^{\frac{2(n+4)}{n-2}}}^{\frac{n-4}{n-2}} \|e^{it\Delta^2} |\nabla| u_0\|_{L^\infty L^{\frac{2n}{n-2}}}^{\frac{2}{n-2}} \\ &\lesssim \|u_0\|_{\dot{H}^2}^{\frac{n-4}{n-2}} \|e^{it\Delta^2} |\nabla| u_0\|_{L^\infty \dot{H}^1}^{\frac{2}{n}} \|e^{it\Delta^2} |\nabla| u_0\|_{L^\infty \dot{B}_{2,\infty}^1}^{\frac{4}{n(n-2)}} \\ &\lesssim \|u_0\|_{\dot{H}^2}^{\frac{n^2-2n-4}{n(n-2)}} \|u_0\|_{\dot{B}_{2,\infty}^s}^{\frac{4}{n(n-2)}}, \end{aligned} \quad (5.4)$$

where for $s = 1, 2$, $\dot{B}_{2,\infty}^s$ is a standard homogeneous Besov space. Now, thanks to (5.4), we may follow the analysis in Bahouri and Gerard [2] and Keraani [21]. In the following, we call scale-core a sequence (h_k, t_k, x_k) such that for every k , $h_k > 0$, $t_k \in \mathbb{R}$ and $x_k \in \mathbb{R}^n$. Mimicking the proof in Keraani [21] we obtain that for $(v_k)_k$ a bounded sequence in \dot{H}^2 , there exists a sequence $(V^\alpha)_\alpha$ in \dot{H}^2 , and scale-cores $(h_k^\alpha, t_k^\alpha, x_k^\alpha)$ such that for any $\alpha \neq \beta$,

$$\left| \log \frac{h_k^\alpha}{h_k^\beta} \right| + (h_k^\alpha)^4 |t_k^\alpha - t_k^\beta| + h_k^\alpha |x_k^\alpha - x_k^\beta| \rightarrow +\infty \quad (5.5)$$

as $k \rightarrow +\infty$, with the property that, up to a subsequence, for any $A \geq 1$,

$$v_k = \sum_{\alpha=1}^A g(h_k^\alpha, x_k^\alpha) \left(e^{-i(h_k^\alpha)^4 t_k^\alpha \Delta^2} V^\alpha \right) + w_k^A \quad (5.6)$$

for all k , where $w_k^A \in \dot{H}^2$ for all k and A , and

$$\lim_{A \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \|e^{it\Delta^2} w_k^A\|_Z = 0. \quad (5.7)$$

Moreover, we have the following estimates:

$$\begin{aligned} \|e^{it\Delta^2} v_k\|_{\dot{H}^2}^{\frac{2(n+4)}{n-4}} &= \sum_{\alpha=1}^{+\infty} \|e^{it\Delta^2} V^\alpha\|_{\dot{H}^2}^{\frac{2(n+4)}{n-4}} + o(1) \quad \text{and,} \\ E(v_k) &= \sum_{\alpha=1}^A E(e^{-i(h_k^\alpha)^4 t_k^\alpha \Delta^2} V^\alpha) + \|w_k^A\|_{\dot{H}^2}^2 + o(1) \end{aligned} \quad (5.8)$$

for all k , where $o(1) \rightarrow 0$ as $k \rightarrow +\infty$. Let $(V, (h_k)_k, (t_k)_k, (x_k)_k)$ be such that $V \in \dot{H}^2$, and $(h_k, t_k, x_k) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ is a scale-core such that $h_k^4 t_k$ has a limit $l \in [-\infty, +\infty]$ as $k \rightarrow +\infty$. We say that U is the nonlinear profile associated to $(V, (h_k)_k, (t_k)_k, (x_k)_k)$ if U is a solution of (1.1) defined on a neighborhood of $-l$, and

$$\|U(-h_k^4 t_k) - e^{-ih_k^4 t_k \Delta^2} V\|_{\dot{H}^2} \rightarrow 0$$

as $k \rightarrow +\infty$. Using the analysis in Pausader [28], it is easily seen that a nonlinear profile always exists and is unique. Besides if

$$E(U) = \lim_k E(e^{-ih_k^4 t_k \Delta^2} V) \quad (5.9)$$

is such that $E(U) < E_{max}$, then the associated nonlinear profile U is globally defined, and

$$\|U\|_{\dot{S}^2(\mathbb{R})} \lesssim_{E(U)} 1.$$

Now, we enter more specifically into the proof of Theorem 5.1. A consequence of Proposition 3.3 is that there exists a sequence of nonlinear solutions u_k such that $E(u_k) < E_{max}$, $E(u_k) \rightarrow E_{max}$, and

$$\|u_k\|_{Z(-\infty, 0)}, \|u_k\|_{Z(0, +\infty)} \rightarrow +\infty. \quad (5.10)$$

We let $((h_k^\alpha)_k, (t_k^\alpha)_k, (x_k^\alpha)_k) = (\mathbf{h}^\alpha, \mathbf{z}^\alpha)$, V^α , and \mathbf{w}^A be given by (5.6) applied to the sequence $(v_k = u_k(0))_k$. Passing to subsequences, and using a diagonal extraction argument, we can assume that, for all α , $(h_k^\alpha)^4 t_k^\alpha$ has a limit in $[-\infty, \infty]$. We let U^α be the nonlinear profile associated to $(V^\alpha, \mathbf{h}^\alpha, \mathbf{z}^\alpha)$. Suppose first that there exists α such that $0 < E(U^\alpha) < E_{max}$. Then, applying (5.8) and (5.9), we see that there exists $\varepsilon > 0$ such that for any β , $E(U^\beta) < E_{max} - \varepsilon$, and we get that all the nonlinear profiles are globally defined. Letting $W_k^A(t) = e^{it\Delta^2} w_k^A$, we remark that

$$p_k^A = \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A$$

satisfies (3.7) with

$$e = e_k^A = f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A\right) - \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)$$

and initial data $p_k^A(0) = u_k(0) + o_A(1)$, where $f(x) = |x|^{\frac{8}{n-4}} x$. First, we claim that

$$\limsup_k \left\| \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \right\|_Z \lesssim_{E_{max}, \varepsilon} 1 \quad (5.11)$$

independently of A . Indeed, we remark that when $(h_k^\alpha, t_k^\alpha, x_k^\alpha)$ and $(h_k^\beta, t_k^\beta, x_k^\beta)$ satisfy (5.5), then for any u, v with finite Z -norm, there holds that

$$\left\| \tau_{(h_k^\beta, t_k^\beta, x_k^\beta)} v \right\|^{\frac{n+12}{n-4}} \tau_{(h_k^\alpha, t_k^\alpha, x_k^\alpha)} u \Big\|_{L^1(\mathbb{R}, L^1)} \rightarrow 0 \quad (5.12)$$

as $k \rightarrow +\infty$, where $\tau_{(h_k, t_k, x_k)}$ is as in (2.10). Now, since Λ is sublinear around 0, and bounded on $[0, E_{max} - \varepsilon]$, using (5.8) and (5.12), we get that

$$\begin{aligned} \limsup_k \left\| \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \right\|_Z &= \left(\sum_{\alpha=1}^A \|U^\alpha\|_Z^{\frac{2(n+4)}{n-4}} \right)^{\frac{n-4}{2(n+4)}} \\ &\lesssim \left(\sum_{\alpha=1}^A \Lambda(E(U^\alpha)) \right)^{\frac{n-4}{2(n+4)}} \\ &\lesssim_{E_{max}, \varepsilon} \left(\sum_{\alpha=1}^A E(U^\alpha) \right)^{\frac{n-4}{2(n+4)}} \\ &\lesssim_{E_{max}, \varepsilon} 1. \end{aligned}$$

Using again (5.12), we get that

$$\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) - \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \right\|_{L^2(\mathbb{R}, L^2)} = o_A(1) \quad (5.13)$$

as $k \rightarrow +\infty$. On the other hand, using the blow-up criterion in Pausader [28, Proposition 2.6.], and the bound $\|U^\alpha\|_Z \leq \Lambda(E(U^\alpha)) \leq \Lambda(E_{max} - \varepsilon)$, we get that, for any α ,

$$\|U^\alpha\|_M \lesssim_{E_{max}, \varepsilon} 1.$$

Using the Leibnitz and chain rules for fractional derivative in Kato [17] and Visan [40, Appendix A], we obtain that

$$\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) - \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \right\|_{L^2(\mathbb{R}, \dot{H}^{\frac{n+8}{n+4}, \frac{2n(n+4)}{n^2+6n+16}})} \lesssim_{A, E_{max}, \varepsilon} 1. \quad (5.14)$$

Interpolating between (5.13) and (5.14), we get that

$$\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) - \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \right\|_N = o_A(1). \quad (5.15)$$

Now, we claim that, letting $s_k^A = \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha$, there holds that

$$\limsup_k \|s_k^A\|_M \lesssim_{E_{max}, \varepsilon} 1, \quad (5.16)$$

independently of A . Indeed, s_k^A satisfies the equation

$$i\partial_t s_k^A + \Delta^2 s_k^A + \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) = 0,$$

with initial data

$$s_k^A(0) = \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha(0) = \sum_{\alpha=1}^A g_{(h_k^\alpha, x_k^\alpha)} e^{-i(h_k^\alpha)^4 t_k^\alpha \Delta^2} V^\alpha + o_A(1),$$

and consequently (5.8) and (5.9) give that

$$\|s_k^A(0)\|_{\dot{H}^2}^2 \leq 2E(s_k^A(0)) \lesssim_{E_{max}} 1 + o_A(1).$$

Using the Strichartz estimates (3.2), (5.11) and (5.15), we get that

$$\begin{aligned}
\|s_k^A\|_M &\lesssim \|s_k^A(0)\|_{\dot{H}^2} + \left\| \sum_{\alpha=1}^A f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \right\|_N \\
&\lesssim E(s_k^A(0))^{\frac{1}{2}} + o_A(1) + \left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) \right\|_N \\
&\lesssim_{E_{max}} 1 + o_A(1) + \|s_k^A\|_Z^{\frac{8}{n-4}} \|s_k^A\|_W \\
&\lesssim_{E_{max}} 1 + o_A(1) + \|s_k^A\|_Z^{\frac{8}{n-4}} \|s_k^A\|_Z^{\frac{1}{2}} \|s_k^A\|_M^{\frac{1}{2}} \\
&\lesssim_{E_{max}, \varepsilon} 1 + o_A(1) + \|s_k^A\|_M^{\frac{1}{2}} \\
&\lesssim_{E_{max}, \varepsilon} 1 + o_A(1)
\end{aligned} \tag{5.17}$$

and (5.17) proves (5.16). Independently,

$$\begin{aligned}
&\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A\right) - f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) \right\|_{L^2(\mathbb{R}, L^2)} \\
&\lesssim \|W_k^A\|_Z \left(\|W_k^A\|_Z^{\frac{8}{n-4}} + \left\| \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \right\|_Z^{\frac{8}{n-4}} \right) \\
&\lesssim_{E_{max}, \varepsilon} \|W_k^A\|_Z \left(\|W_k^A\|_Z^{\frac{8}{n-4}} + 1 \right) \\
&\lesssim_{E_{max}, \varepsilon} \|W_k^A\|_Z
\end{aligned} \tag{5.18}$$

and again, using (5.16) and the product and Leibnitz rules for fractional derivatives, we get that

$$\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A\right) - f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) \right\|_{L^2(\mathbb{R}, \dot{H}^{\frac{n+8}{n+4}, \frac{2n(n+4)}{n^2+6n+1}})} \lesssim_{E_{max}, \varepsilon} 1. \tag{5.19}$$

Interpolating between (5.18) and (5.19), we obtain that

$$\left\| f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A\right) - f\left(\sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\right) \right\|_N \lesssim_{E_{max}, \varepsilon} \|W_k^A\|_Z^{\frac{4}{n+8}} \tag{5.20}$$

and (5.7), (5.15) and (5.20) show that

$$\limsup_k \|e_k^A\|_N = o(1) \tag{5.21}$$

as $A \rightarrow +\infty$. Independently,

$$\begin{aligned}
\|p_k^A\|_W &\leq \left\| \sum_{\alpha=1}^A \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \right\|_W + \|W_k^A\|_W \\
&\lesssim_{E_{max}, \varepsilon} 1 + o_A(1).
\end{aligned} \tag{5.22}$$

Now using Proposition 3.3, (5.21) and (5.22), since $p_k^A(0) = u_k(0) + o_A(1)$, we get that

$$\begin{aligned} \limsup_k \|u_k\|_Z^{\frac{2(n+4)}{n-4}} &\lesssim \lim_{A \rightarrow +\infty} \limsup_k \|p_k^A\|_Z^{\frac{2(n+4)}{n-4}} \\ &\lesssim \sum_{\alpha} \|U^{\alpha}\|_Z^{\frac{2(n+4)}{n-4}} \lesssim_{E_{max}, \varepsilon} \sum_{\alpha} E(U^{\alpha}) \lesssim_{E_{max}, \varepsilon} 1 \end{aligned}$$

and this contradicts (5.10). Now, suppose that for all α , we have that $V^{\alpha} = 0$. Then Strichartz estimates (3.2) and (5.8) give that

$$\begin{aligned} \|e^{it\Delta^2} u_k(0)\|_W &\leq \|e^{it\Delta^2} u_k(0)\|_M^{\frac{1}{2}} \|e^{it\Delta^2} u_k(0)\|_Z^{\frac{1}{2}} \\ &\lesssim E_{max}^{\frac{1}{2}} \|e^{it\Delta^2} u_k(0)\|_Z^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

as $k \rightarrow +\infty$, and Proposition 3.2 gives that $\|u_k\|_Z \rightarrow 0$, which contradicts (5.10). Consequently, we know that there exists a scale core (h_k, t_k, y_k) , and $V \in \dot{H}^2$ such that

$$u_k(0) = g_{(h_k, y_k)} e^{-it_k h_k^4 \Delta^2} V + w_k,$$

where $E(w_k) \rightarrow 0$. Now, up to passing to a subsequence, we can assume that $t_k h_k^4 \rightarrow l \in [-\infty, +\infty]$. If $l \in \mathbb{R}$, then, replacing V by $e^{-il\Delta^2} V$, we can assume that $l = 0$, and changing slightly w_k , we can assume that for any k , $t_k = 0$. We then get that $u_k(0) = g_{(h_k, y_k)} V + o(1)$ in \dot{H}^2 , and in particular $E(V) = E_{max}$. Otherwise, by time reversal symmetry, we can assume that $l = -\infty$, and then, we find that

$$\begin{aligned} \|e^{it\Delta^2} u_k(0)\|_{Z([0, +\infty))} &\leq \|\tau_{(h_k, t_k, y_k)}(e^{it\Delta^2} V)\|_{Z([0, +\infty))} + \|w_k\|_{Z([0, +\infty))} \\ &\leq \|e^{it\Delta^2} V\|_{Z([-h_k^4 t_k, +\infty))} + o(1) \\ &= o(1), \end{aligned}$$

and by standard developements, we get that, for k sufficiently large, $\|u_k\|_{Z(\mathbb{R}_+)}$ remains bounded. Once again, this contradicts (5.10). Let U be the maximal nonlinear solution of (1.1) with initial data V , defined on $I = (-T_*, T^*)$. Suppose, for example that $T^* = +\infty$, and that $\|U\|_{Z(\mathbb{R}_+)} < +\infty$. Then, using Proposition 3.3 on \mathbb{R}_+ with $v = U$, and $u = \tau_{(h_k^{-1}, 0, -y_k)} u_k$, we see that $\|u_k\|_{Z(\mathbb{R}_+)}$ is bounded uniformly in k , which is a contradiction with (5.10). Consequently, we have that

$$\|U\|_{Z(0, T^*)} = \|U\|_{Z(-T_*, 0)} = +\infty$$

and $E(U) = E_{max}$. Now, we prove the compactness property of U . In the sequel, we let $N_{min} > 0$ be sufficiently small so that $\|u\|_{\dot{H}^2} \leq N_{min}$ implies $E(u) < E_{max}/4$. Proceeding as above, it is easily proved by contradiction that for any $\varepsilon > 0$, there exist t_1, \dots, t_j , $j = j(\varepsilon)$, such that for any time $t \in (-T_*, T^*)$, there exist $i = i(t)$, and $g(t) = g_{(h(t), y(t))}$ with the property that $\|u(t_i) - g(t)u(t)\|_{\dot{H}^2} \leq \varepsilon$. Let us apply this with $\varepsilon = N_{min}$. We get a function $g(t) = g_{(h(t), y(t))}$, and a finite set of times t_1, \dots, t_j such that for any t , there exists i satisfying

$$\|u(t_i) - g(t)u(t)\|_{\dot{H}^2} \leq N_{min}.$$

We claim that $K = \{g(t)u(t) : t \in (-T_*, T^*)\}$ is precompact in \dot{H}^2 . Suppose by contradiction that this is not true. Then, there exist $\varepsilon > 0$, and a sequence s_k such that for any k and p ,

$$\|g(s_k)u(s_k) - g(s_p)u(s_p)\|_{\dot{H}^2} > \varepsilon. \quad (5.23)$$

According to what we said above, and passing to a subsequence, we can assume that there exist two times \bar{t}, \bar{t}' , and a sequence $g'_k = g_{(h'_k, y'_k)}$ such that, for any k ,

$$\begin{aligned} \|u(\bar{t}) - g(s_k)u(s_k)\|_{\dot{H}^2} &< N_{min}, \text{ and} \\ \|u(\bar{t}') - g'_k u(s_k)\|_{\dot{H}^2} &< \frac{\varepsilon}{4}. \end{aligned} \quad (5.24)$$

Passing to a subsequence, it is easily seen that $(h'_k)^{-1}h(s_k)$ remains in a compact subset of $(0, \infty)$ and that $y(s_k) - h(s_k)^{-1}h'_k y'_k$ remains in a compact subset of \mathbb{R}^n . Hence, up to considering a subsequence, we can find g_∞ such that $g(s_k)(g'_k)^{-1} \rightarrow g_\infty$ strongly. Now, using (5.24) and the fact that $g_{(h,y)}$ is an isometry on \dot{H}^2 for all (h, y) , we get that

$$\begin{aligned} &\|g(s_k)u(s_k) - g(s_{k+1})u(s_{k+1})\|_{\dot{H}^2} \\ &\leq \|g(s_k)u(s_k) - g_\infty u(\bar{t}')\|_{\dot{H}^2} + \|g_\infty u(\bar{t}') - g(s_{k+1})u(s_{k+1})\|_{\dot{H}^2} \\ &\leq \|g'_k u(s_k) - g'_k g(s_k)^{-1} g_\infty u(\bar{t}')\|_{\dot{H}^2} + \|g'_{k+1} u(s_{k+1}) - g'_{k+1} g(s_{k+1})^{-1} g_\infty u(\bar{t}')\|_{\dot{H}^2} \\ &\leq \frac{\varepsilon}{2} + o(1). \end{aligned}$$

Clearly, this contradicts (5.23) and proves the compactness property of K . The remaining part follows the line of the work in Tao, Visan and Zhang [37] and Killip, Tao and Visan [23]. However, in order to obtain a low-to-high cascade (instead of a high-to-low cascade), we make the following slight modification. We use the notations in Killip, Tao and Visan [23], except for $h(t) = N(t)^{-1}$. In case $Osc(\kappa)$ is unbounded, instead of a , we introduce the quantity

$$b(t_0) = \inf \left(\frac{h(t_0)}{\inf_{t \geq t_0} h(t)}, \frac{h(t_0)}{\inf_{t \leq t_0} h(t)} \right).$$

Then, if $\sup_{t_0 \in J} b(t_0) = +\infty$, we can find intervals on which the solution presents arbitrarily large relative peak. In particular it becomes possible to find a solution satisfying the low-to-high cascade scenario. Finally, in case $\sup_{t_0 \in J} b(t_0) < +\infty$, the solution has arbitrarily large oscillation, but no relative peak. Mimicking the proof in Killip, Tao and Visan [23], but changing future (resp past)-focusing time into future (resp past)-defocusing time, one can find a solution behaving as in the self-similar case scenario. Theorem 5.1 follows. \square

6. THE SELF-SIMILAR CASE

In this section, we deal with the easiest case in Theorem 5.1, namely, the self-similar-like solution. We prove that it is not consistent with conservation of the energy, compactness up to rescaling, and almost conservation of the local L^2 -norm as expressed in (3.11). More precisely, we prove that the following proposition holds true.

Proposition 6.1. *Let $u \in C(I, \dot{H}^2)$ be a maximal-lifespan solution such that $K = \{g(t)u(t) : t \in I\}$ is precompact in \dot{H}^2 for some function g as in (2.11). If $n = 8$, and $I \neq \mathbb{R}$, then $u = 0$. In particular, the self-similar scenario in Theorem 5.1 does not hold true.*

Proof. Let $u \in C(I, \dot{H}^2)$ be a solution as above, with $I \neq \mathbb{R}$, and let $v(t) = g(t)u(t)$. Without loss of generality, we can assume that $\inf I = 0$ and that $(0, 2) \subset I$. Fix

$0 < t < 1$. First, using Hölder's inequality, we get that, for any $\delta > 0$,

$$\int_{B(-h(t)x(t), \delta)} |u(t, x)|^2 dx \lesssim_{E_{max}} \delta^4. \quad (6.1)$$

Independently, let $x_0 \in \mathbb{R}^n$, $R > \delta > 0$, $D = B(x_0, R) \setminus B(-h(t)x(t), \delta)$, and $D' = B(x(t) + x_0/h(t), R/h(t)) \setminus B(0, \delta/h(t))$. Using Hölder's inequality once again, we get that

$$\begin{aligned} \int_D |u(t, x)|^2 dx &= h(t)^4 \int_{D'} |v(t, x)|^2 dx \\ &\leq h(t)^4 \left(\int_{|x| \geq \frac{\delta}{h(t)}} |v(t, x)|^4 dx \right)^{\frac{1}{2}} \left(\int_{B(\frac{x_0}{h(t)} + x(t), \frac{R}{h(t)})} dx \right)^{\frac{1}{2}} \\ &\lesssim \epsilon(\delta/h(t))^{\frac{1}{2}} R^4, \end{aligned} \quad (6.2)$$

where ϵ is given by

$$\epsilon(R) = \sup_{t \in I} \int_{|x| \geq R} |v(t, x)|^4 dx.$$

A consequence of the compactness of K as in Theorem 5.1 is that

$$\epsilon(R) \rightarrow 0, \text{ as } R \rightarrow +\infty. \quad (6.3)$$

Combining (6.1) and (6.2), we get that for any ball B_R of radius $R > \delta$,

$$\int_{B_R} |u(t, x)|^2 dx \lesssim_{E_{max}} \delta^4 + R^4 \epsilon(\delta/h(t))^{\frac{1}{2}}. \quad (6.4)$$

Using almost conservation of local mass, as expressed in (3.11), and (6.4), we get, for any $x_0 \in \mathbb{R}^8$ and any $R > 4$, that the following bound at time 1 holds true

$$\begin{aligned} M(u(1), B(x_0, R))^{\frac{3}{4}} &\lesssim_{E_{max}} \frac{1}{R} + M(u(t), B(x_0, 2R))^{\frac{3}{4}} \\ &\lesssim_{E_{max}} \frac{1}{R} + \left(\delta^4 + R^4 \epsilon(\delta/h(t))^{\frac{1}{2}} \right)^{\frac{3}{4}}, \end{aligned} \quad (6.5)$$

where the local mass is as in (3.10). Letting $t \rightarrow 0$ and using (6.3), and then letting $\delta \rightarrow 0$, we get with (6.5) that

$$M(u(1), B(x_0, R)) \lesssim_{E_{max}} R^{-\frac{4}{3}}. \quad (6.6)$$

Letting $R \rightarrow \infty$ in (6.6), we obtain

$$\|u(1)\|_{L^2} = 0. \quad (6.7)$$

Clearly (6.7) contradicts $u \neq 0$. This proves Proposition 6.1. \square

7. AN INTERACTION MORAWETZ ESTIMATE

To deal with the remaining two scenarii in Theorem 5.1, in which there is no prescribed finite-time blow-up, we need a new ingredient that bounds the amount of nonlinear presence of the solution at a given scale. Natural candidates to achieve this are Morawetz estimates and in our case, interaction Morawetz estimates. In light of Theorem 5.1, we need to work exclusively with \dot{H}^2 -solutions. Interaction Morawetz estimates scale like the $\dot{H}^{\frac{1}{4}}$ -norm. Because of this $7/4$ -difference in scaling, following Colliander, Keel, Staffilani, Takaoka and Tao [7], Ryckman and Visan [32] and Visan [40], we seek for frequency-localized interaction Morawetz estimates. This is the purpose of Sections 7 and 8. In Section 7 we derive an a

priori interaction estimate that applies to all solutions $u \in C(H^2)$, and in Section 8 we use it to obtain a frequency-localized version of these estimates. The frequency localized version applies only to the special \dot{H}^2 -solutions given by Theorem 5.1. We prove here that the following proposition holds true.

Proposition 7.1. *Let $n \geq 7$ and let $u \in C([T_1, T_2], H^2)$ be a solution of (3.1), with forcing term $h \in \dot{S}^2([T_1, T_2]) + \dot{S}^0([T_1, T_2])$. Then the following estimate holds true:*

$$\begin{aligned} & \sum_{j=1}^n \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} \{h, u\}_m(t, y) \frac{(x-y)_j}{|x-y|} \{\partial_j u, u\}_m(t, x) dx dy dt \\ & + \sum_{j=1}^n \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} |u(t, y)|^2 \frac{(x-y)_j}{|x-y|} \{h, u\}_p^j(t, x) dx dy dt \\ & + \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^5} dx dy dt \lesssim \sup_{t=T_1, T_2} \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2, \end{aligned} \quad (7.1)$$

where $\{\cdot, \cdot\}_m$ and $\{\cdot, \cdot\}_p$ are the mass and momentum brackets.

In this proposition, the mass and momentum brackets are defined by

$$\{f, g\}_m = \text{Im}(f\bar{g}), \text{ and } \{f, g\}_p = \text{Re}(f\nabla\bar{g} - g\nabla\bar{f}). \quad (7.2)$$

In addition to Proposition 7.1, in order to exploit the bound given in (7.1), we also prove that the following lemma holds true.

Lemma 7.1. *Assume $n \geq 6$. Then*

$$\| |\nabla|^{-\frac{n-5}{4}} u \|_{L^4} \simeq \| (\sum_N N^{-\frac{n-5}{2}} |P_N u|^2)^{\frac{1}{2}} \|_{L^4} \lesssim \| |\nabla|^{-\frac{n-5}{2}} |u|^2 \|_{L^2}^{\frac{1}{2}}, \quad (7.3)$$

for all $u \in \dot{H}^2$ such that $|\nabla|^{-\frac{3}{2}} |u|^2 \in L^2$, where the summation is over all dyadic numbers.

Proof. The equivalence of norms is classical. We first claim that for any $g \in \mathcal{S}$, and any $n \geq 6$,

$$\| |\nabla|^{-\frac{n-5}{4}} g \|_{L^4} \lesssim \| |\nabla|^{-\frac{n-5}{2}} |g|^2 \|_{L^2}^{\frac{1}{2}}. \quad (7.4)$$

We prove (7.4). Let $\phi(\xi) = |\xi|^{-\frac{n-5}{4}} (\psi(\xi) - \psi(2\xi))$ where ψ is as in (2.4). Using the Cauchy-Schwartz inequality we get that for any dyadic N ,

$$\begin{aligned} & \left(P_N |\nabla|^{-\frac{n-5}{4}} g \right) (x) \\ & = N^{-\frac{n-5}{4}} (g * \mathcal{F}^{-1}(\phi(\xi/N))) (x) \\ & = N^{\frac{3n+5}{4}} \int_{\mathbb{R}^n} g(x-y) \check{\phi}(Ny) dy \\ & \leq N^{\frac{3n+5}{4}} \left(\int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \\ & \lesssim N^{\frac{n+5}{4}} \left(\int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \end{aligned} \quad (7.5)$$

uniformly in N . Since $\phi \in \mathcal{S}$, for any $y \in \mathbb{R}^n$, we get

$$\sum_N (N|y|)^{\frac{n+5}{2}} |\check{\phi}(Ny)| \lesssim \sum_N (N|y|)^{\frac{n+5}{2}} (1 + N|y|)^{-2n} \lesssim 1, \quad (7.6)$$

where the summation is over all dyadic numbers N . Consequently, using (7.5), (7.6) and the fact that $\check{\phi} \in \mathcal{S}$, we get that

$$\begin{aligned} \sum_N |P_N |\nabla|^{-\frac{n-5}{4}} g|^2(x) &\lesssim \sum_N N^{\frac{n+5}{2}} \int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{|g(x-y)|^2}{|y|^{\frac{n+5}{2}}} \left(\sum_N (N|y|)^{\frac{n+5}{2}} |\check{\phi}(Ny)| \right) dy \\ &\lesssim \left(|\nabla|^{-\frac{n-5}{2}} |g|^2 \right)(x), \end{aligned} \quad (7.7)$$

and using the Littlewood-Paley Theorem, (7.7) gives (7.4) for g smooth. Density arguments then give (7.3). This ends the proof of Lemma 7.1. \square

Proof of Proposition 7.1. Since the estimate we want to prove is linear, we can assume that u is smooth and use density arguments to recover the general case. We adopt the convention that repeated indices are summed. Given some real function a , we define the Morawetz action centered at 0 by

$$M_a^0(t) = 2 \int_{\mathbb{R}^n} \partial_j a(x) \operatorname{Im}(\bar{u}(t, x) \partial_j u(t, x)) dx. \quad (7.8)$$

Following the computation in Pausader [28], we get that

$$\begin{aligned} \partial_t M_a^0(t) &= 2 \int_{\mathbb{R}^n} \left(2 \partial_j u \partial_k \bar{u} \partial_{jk} \Delta a - \frac{1}{2} (\Delta^3 a) |u|^2 - 4 \partial_{jk} a \partial_{ik} u \partial_{ij} \bar{u} \right. \\ &\quad \left. + \Delta^2 a |\nabla u|^2 + \partial_j a \{u, h\}_p^j \right) dx. \end{aligned} \quad (7.9)$$

Similarly, we define the Morawetz action centered at y , $M_a^y(t) = M_{a_y}^0(t)$ for $a_y(x) = |x - y|$. Finally, we define the interaction Morawetz action by the following formula:

$$\begin{aligned} M^i(t) &= \int_{\mathbb{R}^n} |u(t, y)|^2 M_a^y(t) dy \\ &= 2 \operatorname{Im} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(t, y)|^2 \frac{x - y}{|x - y|} \nabla u(t, x) \bar{u}(t, x) dx dy \right). \end{aligned} \quad (7.10)$$

We can directly estimate

$$|M^i(t)| \leq \|u\|_{L^\infty L^2}^2 \|u\|_{L^\infty \dot{H}^{\frac{1}{2}}}^2. \quad (7.11)$$

Now, we get an estimate on the variation of M^i by writing that

$$\begin{aligned} \partial_t M^i &= 2 \int_{\mathbb{R}^n} \{u, h\}_m(y) M_a^y dy + 4 \operatorname{Im} \int_{\mathbb{R}^n} \partial_j u(y) \partial_{jk} \bar{u}(y) \partial_k M_a^y dy \\ &\quad + 2 \operatorname{Im} \left(\int_{\mathbb{R}^n} \bar{u}(y) \nabla u(y) \nabla \Delta M_a^y dy \right) + \int_{\mathbb{R}^n} |u(y)|^2 \partial_t M_a^y dy. \end{aligned} \quad (7.12)$$

This gives that

$$\begin{aligned}
\partial_t M^i = & \\
& 4 \int_{\mathbb{R}^n \times \mathbb{R}^n} \operatorname{Im}(\bar{u}(y) \partial_j u(y)) \partial_j^y \Delta(\partial_k^x a(x-y)) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
& + 8 \int_{\mathbb{R}^n \times \mathbb{R}^n} \operatorname{Im}(\partial_i u(y) \partial_{ij} \bar{u}(y)) \partial_j^y (\partial_k^x a(x-y)) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
& + 4 \int_{\mathbb{R}^n \times \mathbb{R}^n} \{u, h\}_m(y) \partial_k^x a(x-y) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
& + 4 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 \partial_{jk}^x (\Delta a(x-y)) \partial_j u(x) \partial_k \bar{u}(x) dx dy \\
& - \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\Delta^3 a(x-y)) |u(x)|^2 dx dy \\
& - 8 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\partial_{jk}^x a(x-y)) \partial_{ik} u(x) \partial_{ij} \bar{u}(x) dx dy \\
& + 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\Delta^2 a(x-y)) |\nabla u(x)|^2 dx dy \\
& + 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 \partial_j^x a(x-y) \{u, h\}_p^j(x) dx dy,
\end{aligned} \tag{7.13}$$

where ∂_j^x denotes derivation with respect to x_j , and ∂_k^y derivation with respect to y_k . Most of the terms in (7.13) have the right sign if we let $a(z) = |z|$. Now we focus on the first two terms in (7.13). In the sequel, we let $z = x - y$. Using the fact that $\operatorname{Re}(AB) = \operatorname{Re}(A) \operatorname{Re}(B) - \operatorname{Im}(A) \operatorname{Im}(B)$, we get the equality:

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \operatorname{Im}(\bar{u}(y) \partial_j u(y)) (\partial_j^y \partial_k^x \Delta a(z)) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
& = -\frac{1}{4} \int_{\mathbb{R}^{2n}} |u(y)|^2 \Delta^3 a(z) |u(x)|^2 dx dy - R((\nabla u \otimes u); (\nabla u \otimes u)),
\end{aligned} \tag{7.14}$$

where we let R be the bilinear form on $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ defined by

$$R((\vec{\alpha} \otimes \beta); (\vec{\gamma} \otimes \delta)) = \operatorname{Re} \int_{\mathbb{R}^{2n}} \alpha_j(y) \bar{\delta}(y) (\partial_{jk}^x \Delta a(z)) \bar{\gamma}_k(x) \beta(x) dx dy. \tag{7.15}$$

For the second term, we proceed as follows:

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \operatorname{Im}(\partial_{ij} \bar{u}(x) \partial_i u(x)) (\partial_j^x \partial_k^y a(z)) \operatorname{Im}(\partial_k u(y) \bar{u}(y)) dx dy \\
& = \frac{1}{4} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \Delta^2 a(z) |u(y)|^2 dx dy + Q((\nabla \partial_i u \otimes u); (\nabla u \otimes \partial_i u)),
\end{aligned} \tag{7.16}$$

where we define the quadratic form Q on $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ by

$$Q((\vec{\alpha} \otimes \beta); (\vec{\gamma} \otimes \delta)) = \operatorname{Re} \int_{\mathbb{R}^{2n}} \alpha_k(x) \bar{\delta}(x) \frac{1}{|z|} \left(\delta_{jk} - \frac{z_j z_k}{|z|^2} \right) \bar{\gamma}_j(y) \beta(y) dy dx. \tag{7.17}$$

As one can check by computing the Fourier transform of its kernel, Q is nonnegative. Hence, applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
|Q((\nabla \partial_i u \otimes u); (\nabla u \otimes \partial_i u))| & \leq |Q((\nabla \partial_i u \otimes u)^2)|^{\frac{1}{2}} |Q((\nabla u \otimes \partial_i u)^2)|^{\frac{1}{2}} \\
& \leq \frac{1}{2} Q((\nabla \partial_i u \otimes u)^2) + \frac{1}{2} Q((\nabla u \otimes \partial_i u)^2)
\end{aligned} \tag{7.18}$$

and if R and Q are as in (7.15) and (7.17), we observe that

$$\begin{aligned}
Q((\nabla u \otimes \partial_i u)^2) &= Q((\nabla \partial_i u \otimes u)^2) - R((\nabla u \otimes u)^2) \\
&\quad + 2\operatorname{Re} \int_{\mathbb{R}^{2n}} \partial_k u(x) \bar{u}(x) (\partial_{ijk}^x a(z)) \partial_{ij} \bar{u}(y) u(y) dx dy \\
&= Q((\nabla \partial_i u \otimes u)^2) + R((\nabla u \otimes u)^2) \\
&\quad + \operatorname{Re} \int_{\mathbb{R}^{2n}} |u(x)|^2 (\partial_{ij}^x \Delta a(z)) \partial_i \bar{u}(y) \partial_j u(y) dx dy.
\end{aligned} \tag{7.19}$$

Consequently, applying (7.14), (7.16), (7.18) and (7.19), we get that

$$\begin{aligned}
&4 \int_{\mathbb{R}^n \times \mathbb{R}^n} \operatorname{Im}(\bar{u}(y) \partial_j u(y)) \partial_j^y \Delta(\partial_k^x a(x-y)) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
&+ 8 \int_{\mathbb{R}^n \times \mathbb{R}^n} \operatorname{Im}(\partial_i u(y) \partial_{ij} \bar{u}(y)) \partial_j^y (\partial_k^x a(x-y)) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
&\leq - \int_{\mathbb{R}^{2n}} |u(y)|^2 (\Delta^3 a(z)) |u(x)|^2 dx dy + 8Q((\nabla \partial_i u \otimes u)^2) \\
&+ 2 \int_{\mathbb{R}^{2n}} |u(y)|^2 (\Delta^2 a(z)) |\nabla u(x)|^2 dx dy \\
&+ 4\operatorname{Re} \int_{\mathbb{R}^{2n}} |u(x)|^2 (\partial_{ij}^x \Delta a(z)) \partial_i \bar{u}(y) \partial_j u(y) dx dy.
\end{aligned} \tag{7.20}$$

Now, for $e \in \mathbb{R}^n$ a vector, and u a function, we define

$$\nabla_e u = (e \cdot \nabla u) \frac{e}{|e|^2}, \text{ and, } \nabla_e^\perp u = \nabla u - \nabla_e u.$$

Then, applying the Cauchy-Schwartz inequality, we get that

$$\begin{aligned}
&Q((\nabla \partial_i u, u)^2) \\
&= \int_{\mathbb{R}^{2n}} \partial_{ij} \bar{u}(x) u(x) \frac{1}{|x-y|} \left(\delta_{jk} - \frac{(x-y)_j (x-y)_k}{|x-y|^2} \right) \partial_{ik} u(y) \bar{u}(y) dx dy \\
&= \int_{\mathbb{R}^{2n}} \frac{[u(x) \nabla_{x-y}^\perp \partial_i u(y)] \cdot [\nabla_{x-y}^\perp \partial_i \bar{u}(x) \bar{u}(y)]}{|x-y|} dx dy \\
&\leq \int_{\mathbb{R}^{2n}} |u(x)|^2 \frac{1}{|x-y|} |\nabla_{x-y}^\perp \partial_i u(y)|^2 dx dy \\
&\leq \int_{\mathbb{R}^{2n}} |u(x)|^2 \frac{1}{|x-y|} \left(\delta_{jk} - \frac{(x-y)_j (x-y)_j}{|x-y|^2} \right) \partial_{ik} \bar{u}(y) \partial_{ij} u(y) dx dy.
\end{aligned} \tag{7.21}$$

Finally, (7.13), (7.20), and (7.21) give

$$\begin{aligned}
\partial_t M^i &\leq -2 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\Delta^3 a(x-y)) |u(x)|^2 dx dy \\
&\quad + 4 \int_{\mathbb{R}^n \times \mathbb{R}^n} \{u, h\}_m(y) \partial_k^x a(x-y) \operatorname{Im}(\partial_k u(x) \bar{u}(x)) dx dy \\
&\quad + 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 \partial_j^x a(x-y) \{u, h\}_p^j(x) dx dy \\
&\quad + 8 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\partial_{jk}^x \Delta a(x-y)) \partial_j u(x) \partial_k \bar{u}(x) dx dy \\
&\quad + 4 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(y)|^2 (\Delta^2 a(x-y)) |\nabla u(x)|^2 dx dy .
\end{aligned} \tag{7.22}$$

Let $T1$ and $T2$ be the last two terms in (7.22). Then

$$\begin{aligned}
&\frac{1}{4(n-1)} (T1 + T2) \\
&= - \int_{\mathbb{R}^{2n}} \frac{|u(y)|^2}{|x-y|^3} \left((n-1) \delta_{jk} - 6 \frac{(x-y)_j (x-y)_k}{|x-y|^2} \right) \partial_j u(x) \partial_k \bar{u}(x) dx dy
\end{aligned}$$

which is nonpositive when $n \geq 7$. Finally, (7.22) and this remark give (7.1). \square

8. A FREQUENCY-LOCALIZED INTERACTION MORAWETZ ESTIMATE

The preceding interaction Morawetz estimate is ill-suited for \dot{H}^2 -solutions. In order to exploit such an estimate in the context of \dot{H}^2 -solutions, we need to localize it at high frequencies. The difficulty then is to deal with an inequality that scales like the $\dot{H}^{\frac{7}{4}}$ -norm, while using only bounds that scale like the \dot{H}^2 -norm. To overcome this difference of $7/4$ derivatives, we split the solution into high and low frequencies and develop an intricate bootstrap argument to get the inequality. This is made possible because we restrict ourselves to the case of the special solutions obtained in Theorem 5.1. More precisely, we prove that the following proposition holds true.

Proposition 8.1. *Let $n = 8$. Let $u \in C(I, \dot{H}^2)$ be a maximal lifespan-solution of (1.1) such that $K = \{g(t)u(t) : t \in I\}$ is precompact in \dot{H}^2 and such that $\forall t \in I$, $h(t) \leq h(0) = 1$. Then, for any sufficiently small $\varepsilon > 0$,*

$$\begin{aligned}
&\| |\nabla|^{-\frac{3}{2}} |P_{\geq 1} u|^2 \|_{L^2(I, L^2)} \lesssim \varepsilon , \\
&\| P_{\geq 1} u \|_{\dot{S}^{-\frac{3}{2}}(I)} \lesssim \varepsilon , \text{ and } \| P_{\leq 1} u \|_{\dot{S}^2(I)} \lesssim \varepsilon
\end{aligned} \tag{8.1}$$

up to replacing u by $g_{(N,0)} u$ for some N .

Proof. We fix $\varepsilon > 0$ sufficiently small to be chosen later on. We remark that for N a dyadic number and for all time,

$$\| P_{\leq N} g^{-1}(t) (g(t)u(t)) \|_{\dot{H}^2} = \| P_{\leq Nh(t)} (g(t)u(t)) \|_{\dot{H}^2}. \tag{8.2}$$

Hence, by compactness of K , and since $h \leq 1$, we have that $\| P_{\leq N} u \|_{L^\infty \dot{H}^2} \rightarrow 0$ as $N \rightarrow 0$. Let N be such that

$$\| P_{\leq \varepsilon^{-4} N} u \|_{L^\infty \dot{H}^2} \leq \frac{\varepsilon}{2}.$$

Replacing K by $Kg_{(\varepsilon^4 N^{-1}, 0)}$, and modifying slightly h , one can assume that

$$\begin{aligned} \|P_{\leq 1}u\|_{L^\infty(I, \dot{H}^2)} &\leq \varepsilon, \text{ and} \\ \|P_{\geq 1}u\|_{L^\infty(I, \dot{H}^s)} &\leq \|P_{1 \leq \cdot < \varepsilon^{-4}}u\|_{L^\infty(I, \dot{H}^2)} + \varepsilon^{4(2-s)} \|P_{\geq \varepsilon^{-4}}u\|_{L^\infty(I, \dot{H}^2)} \\ &\leq \varepsilon, \end{aligned} \quad (8.3)$$

for $s \leq 7/4$. We let

$$J(C) = \{t \geq 0 : \| |\nabla|^{-\frac{3}{2}} |P_{\geq 1}u|^2 \|_{L^2([0, t], L^2)} \leq C\eta\}. \quad (8.4)$$

The first step in the proof is to obtain good Strichartz controls on the high and low-frequency parts of u . In the sequel, we let $u_l = P_{\leq 1}u$, and $u_h = P_{\geq 1}u$. Besides the summations are always over all dyadic numbers, unless otherwise specified. We claim that for $J = J(2)$, we have that

$$\begin{aligned} \| |\nabla|^{-\frac{3}{2}} |P_{\geq 1}u|^2 \|_{L^2(J, L^2)} &\leq 2\eta, \\ \|P_{\leq 1}u\|_{\dot{S}^2(J)} &\lesssim \varepsilon, \text{ and} \\ \|P_{\geq 1}u\|_{\dot{S}^{-\frac{3}{2}}(J)} &\lesssim \eta, \end{aligned} \quad (8.5)$$

provided that $\varepsilon > 0$ is sufficiently small, and that $\varepsilon < \eta$. In the following, all space-time norms are taken on the interval J . Applying the Strichartz estimates (3.3), we get that

$$\begin{aligned} \|P_{\leq 1}u\|_{\dot{S}^2} &\lesssim \|P_{\leq 1}u(0)\|_{\dot{H}^2} + \| |\nabla| P_{\leq 1}(|u_l|^2 u_l) \|_{L^2(J, L^{\frac{8}{5}})} \\ &\quad + \sum_{j=0}^2 \| |\nabla| P_{\leq 1} \mathcal{O}(u_l^j u_h^{3-j}) \|_{L^2(J, L^{\frac{8}{5}})} \\ &\lesssim \varepsilon + \|u_l\|_{\dot{S}^2}^3 + \sum_{j=0}^2 \| |\nabla| P_{\leq 1} \mathcal{O}(u_l^j u_h^{3-j}) \|_{L^2(J, L^{\frac{8}{5}})}. \end{aligned} \quad (8.6)$$

Now, we estimate the terms in the sum. First, using the Bernstein's properties (2.5) and (8.3), we get that

$$\begin{aligned} \| |\nabla| P_{\leq 1} \mathcal{O}(u_l^2 u_h) \|_{L^2 L^{\frac{8}{5}}} &\lesssim \|u_l^2 u_h\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \|u_l\|_{L^4 L^8} \|u_l\|_{L^4 L^8} \|u_h\|_{L^\infty L^{\frac{8}{3}}} \\ &\lesssim \varepsilon \|u_l\|_{\dot{S}^2}^2. \end{aligned} \quad (8.7)$$

For the next term, we remark that if $N \geq 4M$ and $N \geq 8$, then the Fourier support of $P_N u P_M v$ is supported in $\{|\xi| \geq 2\}$, and $P_{\leq 1}(P_N u P_M v) = 0$. Using this remark,

the Bernstein's properties (2.5), (8.3) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
& \| |\nabla| P_{\leq 1} \mathcal{O}(u_l u_h^2) \|_{L^2 L^{\frac{8}{5}}} \\
& \lesssim \| P_{\leq 1} \mathcal{O}(u_l u_h^2) \|_{L^2 L^{\frac{8}{5}}} \\
& \lesssim \sum_{M \leq 1, N \leq 8} \| P_{\leq 1} (P_M u P_N \mathcal{O}(u_h^2)) \|_{L^2 L^{\frac{8}{5}}} \\
& \lesssim \left(\sum_{M \leq 1} \| P_M u \|_{L^\infty L^8} \right) \left(\sum_{N \leq 8} \| P_N \mathcal{O}(u_h^2) \|_{L^2 L^2} \right) \\
& \lesssim \left(\sum_{M \leq 1} M^{-1} \| P_M u \|_{L^\infty L^8}^2 \right)^{\frac{1}{2}} \left(\sum_{N \leq 8} N^{-3} \| P_N \mathcal{O}(u_h^2) \|_{L^2 L^2}^2 \right)^{\frac{1}{2}} \\
& \lesssim \| u_l \|_{L^\infty \dot{H}^2} \| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2} \\
& \lesssim \varepsilon \eta,
\end{aligned} \tag{8.8}$$

where we have used in the last inequalities that since $|\nabla|^{-\frac{3}{2}}$ has a positive kernel, we have that $\| |\nabla|^{-\frac{3}{2}} \mathcal{O}(u_h^2) \|_{L^2 L^2} \leq \| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2}$. We treat the last term similarly as follows, by writing that

$$\begin{aligned}
& \| |\nabla| P_{\leq 1} \mathcal{O}(u_h^3) \|_{L^2 L^{\frac{8}{5}}} \\
& \lesssim \| P_{\leq 1} \mathcal{O}(u_h^3) \|_{L^2 L^1} \\
& \lesssim \sum_{1 \leq N \leq 8, M \leq 32} \| P_{\leq 1} (P_N u_h P_M \mathcal{O}(u_h^2)) \|_{L^2 L^1} \\
& + \sum_{N \geq 8, 4N \geq M \geq N/4} \| P_{\leq 1} (P_N u_h P_M \mathcal{O}(u_h^2)) \|_{L^2 L^1} \\
& \lesssim \left(\sum_M M^3 \| P_M u_h \|_{L^\infty L^2}^2 \right)^{\frac{1}{2}} \left(\sum_M M^{-3} \| P_M |u_h|^2 \|_{L^2 L^2}^2 \right)^{\frac{1}{2}} \\
& \lesssim \| u_h \|_{L^\infty H^{\frac{7}{4}}} \| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2} \\
& \lesssim \varepsilon \eta.
\end{aligned} \tag{8.9}$$

Finally, we get with (8.6)–(8.9) that

$$\| u_l \|_{\dot{S}^2} \lesssim \| P_{\leq 1} u \|_{\dot{S}^2} \lesssim \varepsilon + \eta \varepsilon + \varepsilon \| u_l \|_{\dot{S}^2}^2 + \| u_l \|_{\dot{S}^2}^3 \tag{8.10}$$

and this proves the second inequality in (8.5) with u_l instead of $P_{\leq 1} u$ if $\varepsilon > 0$ is sufficiently small. Using again (8.10), we get the second inequality in (8.5). Now we turn to the control on u_h . Still using the Strichartz estimates (3.3) and Sobolev's inequality, we get that

$$\begin{aligned}
\| u_h \|_{\dot{S}^{-\frac{3}{2}}} & \lesssim \| u_h(0) \|_{\dot{H}^{-\frac{3}{2}}} + \sum_{j=0}^3 \| |\nabla|^{-\frac{5}{2}} P_{\geq 1} \mathcal{O}(u_h^j u_l^{3-j}) \|_{L^2 L^{\frac{8}{5}}} \\
& \lesssim \varepsilon + \sum_{j=2,3} \| P_{\geq 1} \mathcal{O}(u_h^j u_l^{3-j}) \|_{L^2 L^{\frac{16}{15}}} \\
& + \sum_{j=0,1} \| |\nabla|^{-\frac{5}{2}} P_{\geq 1} \mathcal{O}(u_h^j u_l^{3-j}) \|_{L^2 L^{\frac{8}{5}}}.
\end{aligned} \tag{8.11}$$

By convolution estimate, letting $c_N = N^{-\frac{3}{4}}|P_N u_h|$, we get that

$$\begin{aligned} ||u_h|^2 u_h| &\lesssim \left| \sum_{M_1 \geq M_2 \geq M_3} \mathcal{O}(P_{M_1} u_h P_{M_2} u_h P_{M_3} u_h) \right| \\ &\lesssim \left| \sum_{M_1 \geq M_2 \geq M_3} c_{M_3} \left(\frac{M_3}{M_2} \right)^{\frac{3}{4}} c_{M_2} \left(\frac{M_2}{M_1} \right)^{\frac{3}{2}} M_1^{\frac{3}{2}} P_{M_1} u_h \right| \\ &\lesssim \left(\sum_M c_M^2 \right) \left(\sup_M M^{\frac{3}{2}} |P_M u_h| \right). \end{aligned} \quad (8.12)$$

Consequently, using the Bernstein's properties (2.5), (7.3) and (8.12), we get that

$$\begin{aligned} &\|P_{\geq 1}|u_h|^2 u_h\|_{L^2 L^{\frac{16}{15}}} \\ &\lesssim \| |u_h|^2 u_h \|_{L^2 L^{\frac{16}{15}}} \\ &\lesssim \left\| \left(\sum_M M^{-\frac{3}{2}} |P_M u_h|^2 \right)^{\frac{1}{2}} \right\|_{L^4 L^4}^2 \left(\left\| \sup_M |\nabla|^{\frac{3}{2}} P_M u_h \right\|_{L^\infty L^{\frac{16}{7}}} \right) \\ &\lesssim \| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2} \|u_h\|_{L^\infty \dot{H}^2} \\ &\lesssim_{E_{max}} \eta. \end{aligned} \quad (8.13)$$

Note that instead of using the pointwise evaluation of $u_h = \sum P_M u_h$, we can replace u_h by an arbitrary Schwartz function, get the bound, and then use density arguments to recover (8.13). When $j = 2$, we proceed as follows, using Sobolev's inequality, the Bernstein's properties (2.5), (8.3) and the estimate for u_l in (8.5),

$$\begin{aligned} \|\mathcal{O}(u_h^2 u_l)\|_{L^2 L^{\frac{16}{15}}} &\lesssim \|u_l\|_{L^4 L^8} \|u_h\|_{L^4 L^{\frac{16}{7}}} \|u_h\|_{L^\infty L^{\frac{8}{3}}} \\ &\lesssim \varepsilon \|u_l\|_{\dot{S}^2} \| |\nabla|^{-\frac{3}{2}} u_h \|_{L^2 L^{\frac{8}{3}}}^{\frac{1}{2}} \| |\nabla|^{\frac{3}{2}} u_h \|_{L^\infty L^2}^{\frac{1}{2}} \\ &\lesssim \varepsilon^{\frac{5}{2}} \|u_h\|_{\dot{S}^{-\frac{3}{2}}}^{\frac{1}{2}}. \end{aligned} \quad (8.14)$$

When $j = 1$, we proceed similarly to get

$$\begin{aligned} \| |\nabla|^{-\frac{5}{2}} P_{\geq 1} \mathcal{O}(u_l^2 u_h) \|_{L^2 L^{\frac{8}{5}}} &\lesssim \|\mathcal{O}(u_l^2 u_h)\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \|u_l\|_{L^4 L^8}^2 \|u_h\|_{L^\infty L^{\frac{8}{3}}} \\ &\lesssim \varepsilon^3, \end{aligned} \quad (8.15)$$

and finally,

$$\begin{aligned} \| |\nabla|^{-\frac{5}{2}} P_{\geq 1} |u_l|^2 u_l \|_{L^2 L^{\frac{8}{5}}} &\lesssim \| |\nabla| |u_l|^2 u_l \|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \|u_l\|_{\dot{S}^2}^3 \\ &\lesssim \varepsilon^3. \end{aligned} \quad (8.16)$$

Combining (8.11) and (8.13)–(8.16), we get that

$$\begin{aligned} \|u_h\|_{\dot{S}^{-\frac{3}{2}}} &\lesssim \varepsilon + \eta + \varepsilon^3 + \varepsilon^{\frac{5}{2}} \|u_h\|_{\dot{S}^{-\frac{3}{2}}}^{\frac{1}{2}} \\ &\lesssim \eta. \end{aligned}$$

This ends the proof of (8.5). As a consequence of conservation of energy, (8.3), (8.5) and Hardy-Littlewood-Sobolev's inequality, we get the following estimates on

$J = J(2)$. Namely,

$$\begin{aligned} \|u_h\|_{L^{\frac{5}{2}}L^{\frac{5}{2}}} &\lesssim_{E_{max}} \eta^{\frac{4}{5}}, \quad \|u_h\|_{L^3L^{\frac{8}{3}}} \lesssim_{E_{max}} \eta^{\frac{2}{3}}, \quad \|u_h\|_{L^{\frac{9}{2}}L^3} \lesssim_{E_{max}} \eta^{\frac{4}{9}}, \quad \text{and} \\ \| |u_h|^2 * |x|^{-1} \|_{L^3L^{24}} &\lesssim \|u_h\|_{L^6L^{\frac{24}{11}}}^2 \lesssim_{E_{max}} \varepsilon^{\frac{4}{3}} \eta^{\frac{2}{3}} \end{aligned} \quad (8.17)$$

Now that we have good Strichartz control on the high and low frequencies, we can control the error terms arising in the frequency-localized interaction Morawetz estimates. First, we treat the terms arising from the mass bracket. We claim that on $J = J(2)$, as defined above, we have that

$$\int_J \int_{\mathbb{R}^{2n}} \{P_{\geq 1}(|u|^2 u), u\}_m(t, y) \frac{(x-y)_j}{|x-y|} \{\partial_j u, u\}_m(t, x) dx dy \lesssim \varepsilon^2 \eta^2. \quad (8.18)$$

Exploiting cancellations, we write

$$\begin{aligned} \{P_{\geq 1}(|u|^2 u), u_h\}_m &= \{P_{\geq 1}(|u|^2 u - |u_h|^2 u_h), u_h\}_m \\ &\quad - \{P_{< 1}(|u_h|^2 u_h), u_h\}_m + \{|u_h|^2 u_h, u_h\}_m. \end{aligned} \quad (8.19)$$

The last term in the right-hand side of (8.19) vanishes. Using the Bernstein's properties (2.5), (8.3) and (8.17), we get that

$$\begin{aligned} &\left| \int_J \int_{\mathbb{R}^{2n}} \operatorname{Im}(\partial_k u_h(x) \bar{u}_h(x)) \frac{(x-y)_k}{|x-y|} \{P_{< 1}|u_h|^2 u_h, u_h\}_m(y) dx dy dt \right| \\ &\lesssim \|u_h\|_{L^\infty L^2} \|\nabla u_h\|_{L^\infty L^2} \int_J |(P_{< 1}|u_h|^2 u_h) u_h| dx dt \\ &\lesssim \varepsilon^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|P_{< 1}|u_h|^2 u_h\|_{L^{\frac{3}{2}} L^{\frac{8}{5}}} \\ &\lesssim \varepsilon^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \| |u_h|^2 u_h \|_{L^{\frac{3}{2}} L^1} \\ &\lesssim \varepsilon^2 \|u_h\|_{L^3 L^{\frac{8}{3}}}^3 \|u_h\|_{L^\infty L^4} \\ &\lesssim \varepsilon^2 \eta^2. \end{aligned} \quad (8.20)$$

As for the first term in (8.19), using (8.3), we get that

$$\begin{aligned} &\left| \int_J \int_{\mathbb{R}^{2n}} \{\partial_k u_h, u_h\}_m(x) \frac{(x-y)_k}{|x-y|} \{P_{\geq 1}(|u|^2 u - |u_h|^2 u_h), u_h\}_m(y) dx dy dt \right| \\ &\lesssim \sum_{j=0}^2 \|u_h\|_{L^\infty H^1}^2 \int_J \left| \left(P_{\geq 1} \mathcal{O}(u_h^j u_l^{3-j}) \right) u_h \right| dx dt \end{aligned} \quad (8.21)$$

and, using the Bernstein's properties (2.5), (8.5), and (8.17), we obtain

$$\begin{aligned} \int_J |P_{\geq 1} \mathcal{O}(u_h^2 u_l) u_h| dx dt &\lesssim \|u_h\|_{L^{\frac{9}{2}} L^3} \|u_h^2 u_l\|_{L^{\frac{9}{7}} L^{\frac{3}{2}}} \\ &\lesssim \|u_h\|_{L^{\frac{9}{2}} L^3}^3 \|u_l\|_{L^3 L^\infty} \\ &\lesssim \eta^{\frac{4}{3}} \varepsilon. \end{aligned} \quad (8.22)$$

Similarly,

$$\begin{aligned} \int_J |P_{\geq 1} \mathcal{O}(u_h u_l^2) u_h| dx dt &\lesssim \|u_h\|_{L^3 L^{\frac{8}{3}}}^2 \|u_l\|_{L^6 L^8}^2 \\ &\lesssim \eta^{\frac{4}{3}} \varepsilon^2. \end{aligned} \quad (8.23)$$

In order to treat the last term, we remark that, in view of the Fourier support, if $M_1, M_2, M_3 \leq 1/8$, then $P_{\geq 1}(P_{M_1} u P_{M_2} u P_{M_3} u) = 0$. Consequently, letting $c_M =$

$M^2 \|P_M u\|_{L^2 L^4}$ and $d_M = M^2 \|P_M u\|_{L^\infty L^2}$, we get, using again the Bernstein's properties (2.5), (8.3) and (8.5), that

$$\begin{aligned}
& \int_J |(P_{\geq 1}|u_l|^2 u_l) u_h| dx dt \\
& \lesssim \|u_h\|_{L^\infty L^2} \|P_{\geq 1} \sum_{1 \geq M_1 \geq M_2 \geq M_3} P_{M_1} u P_{M_2} u P_{M_3} u\|_{L^1 L^2} \\
& \lesssim \|u_h\|_{L^\infty L^2} \sum_{1 \geq M_1 \geq 1/8, M_1 \geq M_2 \geq M_3} \|P_{M_1} u P_{M_2} u P_{M_3} u\|_{L^1 L^2} \\
& \lesssim \|u_h\|_{L^\infty L^2} \sum_{1 \geq M_1 \geq 1/8, M_1 \geq M_2 \geq M_3} \|P_{M_1}\|_{L^2 L^4} \|P_{M_2} u\|_{L^2 L^4} \|P_{M_3} u\|_{L^\infty L^\infty} \quad (8.24) \\
& \lesssim \|u_h\|_{L^\infty L^2} \left(\sum_{1 \geq M \geq 1/8} \|P_M u\|_{L^2 L^4} \right) \sum_{1 \geq M_2 \geq M_3} c_{M_2} d_{M_3} \left(\frac{M_3}{M_2} \right)^2 \\
& \lesssim \|u_h\|_{L^\infty L^2} \|u_l\|_{\dot{S}^2}^3 \\
& \lesssim \varepsilon^4.
\end{aligned}$$

Combining (8.19)–(8.24), we see that (8.18) holds true. Now, we turn to the last error term, which arises from the momentum bracket. We claim that on $J = J(2)$, we have that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(s, y)|^2 \frac{(x-y)_j}{|x-y|} \{P_{\geq 1}|u|^2 u, u_h\}_p^j(s, x) dx dy ds \right. \\
& \quad \left. - \frac{1}{2} \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(s, y)|^2 |u_h(s, x)|^4}{|x-y|} dx dy ds \right| \quad (8.25) \\
& \lesssim \eta^2 \left(\varepsilon^{\frac{7}{3}} \eta^{-\frac{2}{3}} + \varepsilon^2 + \varepsilon^{\frac{16}{3}} \eta^{-\frac{4}{3}} \right).
\end{aligned}$$

In order to prove (8.25), we decompose

$$\begin{aligned}
\{P_{\geq 1}|u|^2 u, u_h\}_p &= \{|u|^2 u, u\}_p - \{|u_l|^2 u_l, u_l\}_p - \{(|u|^2 u - |u_l|^2 u_l), u_l\}_p \\
&\quad - \{P_{< 1}|u|^2 u, u_h\}_p \\
&= -\frac{1}{2} \nabla (|u|^4 - |u_l|^4) - \{(|u|^2 u - |u_l|^2 u_l), u_l\}_p \\
&\quad - \{P_{< 1}|u|^2 u, u_h\}_p. \quad (8.26)
\end{aligned}$$

Besides, we remark that

$$\{f, g\}_p = \nabla \mathcal{O}(fg) - \mathcal{O}(f \nabla g). \quad (8.27)$$

Now, we estimate

$$\mathcal{R} = \sum_{k=0}^2 \int_J \int_{\mathbb{R}^{2n}} |u_h(s, y)|^2 \frac{(x-y)_j}{|x-y|} \{\mathcal{O}(u_l^k u_h^{3-k}), u_l\}_p^j(s, x) dx dy ds. \quad (8.28)$$

The case $k = 2$ is treated as follows, using (8.27). First

$$\begin{aligned} & \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} \mathcal{O}(u_h u_l^2) \partial_j u_l(x) ds dx dy \right| \\ & \lesssim \left| \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \mathcal{O} \int_{\mathbb{R}^n} (|\nabla|^{-1} u_h) \left(|\nabla| \left(\frac{(x-y)_j}{|x-y|} (u_l^2 \partial_j u_l)(x) \right) \right) dx ds dy \right|. \end{aligned} \quad (8.29)$$

Now, using the boundedness of the Riesz transform and the Bernstein's properties (2.5), we estimate for any $y \in \mathbb{R}^n$,

$$\begin{aligned} & \left\| |\nabla| \left(\frac{(x-y)_j}{|x-y|} (u_l^2 \partial_j u_l)(x) \right) \right\|_{L^{\frac{8}{5}}} \\ & \lesssim \left\| \nabla \left(\frac{(x-y)_j}{|x-y|} (u_l^2 \partial_j u_l)(x) \right) \right\|_{L^{\frac{8}{5}}} \\ & \lesssim \| |x-y|^{-1} \mathbf{1}_{\{|x-y| \leq 1\}} \|_{L^2} \|u_l^2\|_{L^\infty} \|\nabla u_l\|_{L^8} \\ & \quad + \| |x-y|^{-1} \mathbf{1}_{\{|x-y| \geq 1\}} \|_{L^\infty} \|u_l^2\|_{L^4} \|\nabla u_l\|_{L^{\frac{8}{3}}} \\ & \quad + \|\nabla u_l\|_{L^8} \|\partial_j u_l\|_{L^{\frac{8}{3}}} \|u_l\|_{L^8} + \|u_l^2\|_{L^4} \|\nabla^2 u_l\|_{L^{\frac{8}{3}}} \\ & \lesssim \|u_l\|_{L^8}^2 \|\nabla u_l\|_{L^{\frac{8}{3}}}, \end{aligned} \quad (8.30)$$

where $\mathbf{1}_E$ is the characteristic function of the set E . Consequently, using the Bernstein's properties (2.5), (8.3), (8.5) and (8.30), we get that

$$\begin{aligned} & \left| \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \mathcal{O} \int_{\mathbb{R}^n} (|\nabla|^{-1} u_h) \left(|\nabla| \left(\frac{(x-y)_j}{|x-y|} (u_l^2 \partial_j u_l)(x) \right) \right) dx ds dy \right| \\ & \lesssim \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \| |\nabla|^{-1} u_h \|_{L^{\frac{8}{5}}} \| |\nabla| \left(\frac{(x-y)_j}{|x-y|} u_l^2 \partial_j u_l \right) \|_{L^{\frac{8}{5}}} \\ & \lesssim \|u_h\|_{L^\infty L^2}^2 \| |\nabla|^{-\frac{1}{2}} u_h \|_{L^2 L^{\frac{8}{3}}} \|u_l\|_{L^4 L^8}^2 \|\nabla u_l\|_{L^\infty L^{\frac{8}{3}}} \\ & \lesssim \eta \varepsilon^5. \end{aligned} \quad (8.31)$$

Besides, integrating by parts and using (8.5) and (8.17), we finish the analysis of the case $k = 2$ as follows:

$$\begin{aligned} & \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} \mathcal{O}(u_l^3 u_h)(x) dx dy ds \right| \\ & \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^{\frac{5}{2}} L^{\frac{5}{2}}} \|u_l\|_{L^{\frac{45}{4}} L^{\frac{360}{67}}}^3 \\ & \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^{\frac{5}{2}} L^{\frac{5}{2}}} \|u_l\|_{\dot{S}^2}^3 \\ & \lesssim \eta \varepsilon^4. \end{aligned} \quad (8.32)$$

The case $k = 1$ is similar. First, with the Bernstein's properties (2.5), (8.3), (8.5) and (8.17), we obtain that

$$\begin{aligned} & \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} \mathcal{O}(u_h^2 u_l)(x) \partial_j u_l(x) ds dx dy \right| \\ & \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_h^2\|_{L^{\frac{3}{2}} L^{\frac{4}{3}}} \|\nabla u_l\|_{L^3 L^{\frac{24}{5}}} \|u_l\|_{L^\infty L^{24}} \\ & \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_l\|_{L^\infty L^4} \|u_l\|_{\dot{S}^2}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}}^2 \\ & \lesssim \varepsilon^4 \eta^{\frac{4}{3}} \end{aligned} \quad (8.33)$$

and then,

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} \mathcal{O}(u_h^2 u_l^2)(x) ds dx dy \right| \\
& \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^\infty L^2} \|u_h\|_{L^\infty L^{\frac{8}{3}}} \|u_l\|_{L^3 L^{24}}^2 \\
& \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^\infty H^1}^2 \|u_l\|_{L^3 L^{12}}^2 \\
& \lesssim \varepsilon^{\frac{16}{3}} \eta^{\frac{2}{3}}.
\end{aligned} \tag{8.34}$$

Finally for the case $k = 0$, using the Bernstein's properties (2.5), (8.3), (8.5) and (8.17), we write that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} \mathcal{O}(u_h^3)(x) \partial_j u_l(x) ds dx dy \right| \\
& \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_h^3\|_{L^{\frac{3}{2}} L^1} \|\nabla u_l\|_{L^3 L^\infty} \\
& \lesssim \eta^{\frac{4}{3}} \varepsilon^3,
\end{aligned} \tag{8.35}$$

and that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} \mathcal{O}(u_h^3 u_l)(x) ds dx dy \right| \\
& \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^3 L^{\frac{8}{3}}} \|u_h\|_{L^\infty L^4}^2 \|u_l\|_{L^3 L^{12}} \\
& \lesssim \varepsilon^{\frac{7}{3}} \eta^{\frac{4}{3}}.
\end{aligned} \tag{8.36}$$

This finishes the analysis of the second error term in the momentum bracket (8.26), namely \mathcal{R} . Now we turn to the third error term arising from (8.26), i.e.

$$\tilde{\mathcal{R}} = \sum_{k=0}^3 \int_J \int_{\mathbb{R}^{2n}} |u_h(s, y)|^2 \frac{(x-y)_j}{|x-y|} \{P_{<1} \mathcal{O}(u_h^k u_l^{3-k}), u_h\}_p^j(s, x) dx dy ds.$$

We treat the term $k = 0$ using (8.27) as follows. First, we get that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \int_{\mathbb{R}^n} u_h \frac{(x-y)_j}{|x-y|} \partial_j (P_{<1} \mathcal{O}(u_l^3))(x) dx dy ds \right| \\
& = \left| \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \int_{\mathbb{R}^n} (|\nabla|^{-1} u_h) \left(|\nabla| \left(\frac{(x-y)_j}{|x-y|} \partial_j P_{<1} \mathcal{O}(u_l^3) \right) \right) dx dy ds \right| \\
& \leq \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \| |\nabla|^{-1} u_h \|_{L^{\frac{8}{3}}} \| |\nabla| \left(\frac{(x-y)_j}{|x-y|} \partial_j P_{<1} \mathcal{O}(u_l^3) \right) \|_{L^{\frac{8}{5}}} ds dy.
\end{aligned} \tag{8.37}$$

Using the boundedness of the Riesz transform, we see that

$$\begin{aligned}
& \| |\nabla| \left(\frac{(x-y)_j}{|x-y|} \partial_j P_{<1} \mathcal{O}(u_l^3) \right) \|_{L^{\frac{8}{5}}} \\
& \lesssim \| \nabla \left(\frac{(x-y)_j}{|x-y|} \partial_j P_{<1} \mathcal{O}(u_l^3) \right) \|_{L^{\frac{8}{5}}} \\
& \lesssim \| \mathbf{1}_{\{|x-y| \leq 1\}} |x-y|^{-1} \|_{L^2} \| \partial_j u_l \|_{L^8} \| u_l \|_{L^\infty}^2 \\
& + \| \mathbf{1}_{\{|x-y| \geq 1\}} |x-y|^{-1} \|_{L^\infty} \| \partial_j u_l \|_{L^8} \| u_l \|_{L^4}^2 \\
& + \| \nabla \partial_j u_l \|_{L^8} \| u_l \|_{L^4}^2 + \| \nabla u_l \|_{L^8} \| \nabla u_l \|_{L^4} \| u_l \|_{L^4} \\
& \lesssim \| u_l \|_{L^4}^2 \| \nabla u_l \|_{L^8},
\end{aligned} \tag{8.38}$$

and, consequently, using the Bernstein's properties (2.5), (8.3), (8.5) and (8.38) above, we obtain that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \left(\int_{\mathbb{R}^n} u_h(x) \frac{(x-y)_j}{|x-y|} \partial_j (P_{<1} |u_l|^2 u_l)(x) dx \right) dy ds \right| \\
& \lesssim \int_J \int_{\mathbb{R}^n} |u_h(y)|^2 \| |\nabla|^{-1} u_h \|_{L^{\frac{8}{3}}} \| \nabla u_l \|_{L^8} \| u_l \|_{L^4}^2 dy ds \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| |\nabla|^{-\frac{1}{2}} u_h \|_{L^2 L^{\frac{8}{3}}} \| \nabla u_l \|_{L^2 L^8} \| u_l \|_{L^\infty L^4}^2 \\
& \lesssim \eta \varepsilon^5.
\end{aligned} \tag{8.39}$$

As for the other part, using (8.5) and (8.17), we get that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(y)|^2}{|x-y|} P_{<1} \mathcal{O}(u_l^3)(x) u_h(x) dx dy ds \right| \\
& \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \| u_h \|_{L^3 L^{\frac{8}{3}}} \| u_l \|_{L^3 L^{12}} \| u_l \|_{L^\infty L^4}^2 \\
& \lesssim \varepsilon^{\frac{16}{3}} \eta^{\frac{4}{3}}.
\end{aligned} \tag{8.40}$$

Now, we treat the case $k = 1$ using Bernstein property (2.5), (8.3), (8.5) and (8.17) as follows. First we write that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} u_h(x) \partial_j (P_{<1} \mathcal{O}(u_l^2 u_h))(x) dx dy ds \right| \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| u_h \|_{L^3 L^{\frac{8}{3}}} \| u_l^2 u_h \|_{L^{\frac{3}{2}} L^{\frac{8}{5}}} \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| u_h \|_{L^3 L^{\frac{8}{3}}}^2 \| u_l \|_{L^6 L^8}^2 \\
& \lesssim \varepsilon^4 \eta^{\frac{4}{3}},
\end{aligned} \tag{8.41}$$

and then we write that

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(y)|^2}{|x-y|} u_h(x) P_{<1} \mathcal{O}(u_l^2 u_h)(x) dx dy ds \right| \\
& \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \| u_h \|_{L^3 L^{\frac{8}{3}}}^2 \| u_l \|_{L^\infty L^{\frac{48}{5}}}^2 \\
& \lesssim \varepsilon^{\frac{10}{3}} \eta^2.
\end{aligned} \tag{8.42}$$

When $k = 2$, we use the Bernstein's properties (2.5), (8.3), (8.5), and (8.17) to get

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} u_h(x) \partial_j (P_{<1} \mathcal{O}(u_l u_h^2))(x) dx dy ds \right| \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| u_h \|_{L^3 L^{\frac{8}{3}}} \| \partial_j P_{<1} \mathcal{O}(u_l u_h^2) \|_{L^{\frac{3}{2}} L^{\frac{8}{5}}} \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| u_h \|_{L^3 L^{\frac{8}{3}}} \| u_h^2 u_l \|_{L^{\frac{3}{2}} L^{\frac{8}{5}}} \\
& \lesssim \| u_h \|_{L^\infty L^2}^2 \| u_h \|_{L^3 L^{\frac{8}{3}}}^3 \| u_l \|_{L^\infty L^\infty} \\
& \lesssim \varepsilon^3 \eta^2,
\end{aligned} \tag{8.43}$$

and

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(y)|^2}{|x-y|} u_h(x) P_{<1} \mathcal{O}(u_l u_h^2)(x) dx dy ds \right| \\
& \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^3 L^{\frac{8}{3}}} \|P_{<1} \mathcal{O}(u_l u_h^2)\|_{L^3 L^{\frac{12}{7}}} \\
& \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|P_{<1} \mathcal{O}(u_l u_h^2)\|_{L^3 L^1} \\
& \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}}^2 \|u_h\|_{L^\infty L^2} \|u_l\|_{L^\infty L^8} \\
& \lesssim \varepsilon^{\frac{10}{3}} \eta^2.
\end{aligned} \tag{8.44}$$

Finally, the case $k = 3$ is treated as follows using the Bernstein's properties (2.5), (8.3), (8.5), and (8.17)

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} u_h(x) \partial_j (P_{<1} \mathcal{O}(u_h^3))(x) dx dy ds \right| \\
& \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|\nabla P_{<1} \mathcal{O}(u_h^3)\|_{L^{\frac{3}{2}} L^{\frac{8}{3}}} \\
& \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|u_h^3\|_{L^{\frac{3}{2}} L^1} \\
& \lesssim \|u_h\|_{L^\infty L^2}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|u_h\|_{L^{\frac{9}{2}} L^3}^3 \\
& \lesssim \varepsilon^2 \eta^2
\end{aligned} \tag{8.45}$$

and, similarly,

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(y)|^2}{|x-y|} u_h(x) P_{<1} \mathcal{O}(u_h^3)(x) dx dy ds \right| \\
& \lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^3 L^{\frac{8}{3}}} \|P_{<1} \mathcal{O}(u_h^3)\|_{L^3 L^{\frac{12}{7}}} \\
& \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}} \|P_{<1} \mathcal{O}(u_h^3)\|_{L^3 L^1} \\
& \lesssim \|u_h\|_{L^6 L^{\frac{24}{11}}}^2 \|u_h\|_{L^3 L^{\frac{8}{3}}}^2 \|u_h\|_{L^\infty L^{\frac{16}{5}}}^2 \\
& \lesssim \varepsilon^{\frac{10}{3}} \eta^2.
\end{aligned} \tag{8.46}$$

This finishes the analysis of $\tilde{\mathcal{R}}$. The first error term in (8.26) is now easy to treat. Indeed, integrating by parts,

$$\begin{aligned}
& \left| \int_J \int_{\mathbb{R}^{2n}} |u_h(s, y)|^2 \frac{(x-y)}{|x-y|} \nabla (|u|^4 - |u_l|^4 - |u_h|^4)(s, x) dx dy ds \right| \\
& \leq \sum_{k=1}^3 \mathcal{O} \int_J \int_{\mathbb{R}^{2n}} \frac{|u_h(s, y)|^2 |u_h(s, x)|^{4-k} |u_l(s, x)|^k}{|x-y|} dx dy ds \\
& \lesssim (8.32) + (8.34) + (8.36) \\
& \lesssim \varepsilon^{\frac{7}{3}} \eta^{\frac{4}{3}}.
\end{aligned} \tag{8.47}$$

Finally, with (8.26)–(8.47), we obtain (8.25). As a consequence of (7.1), (8.18) and (8.25) on $J = J(2)$, we have that

$$\| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2}^2 \lesssim \eta^2 \left(\varepsilon^{\frac{7}{3}} \eta^{-\frac{2}{3}} + \varepsilon^2 + \varepsilon^{\frac{16}{3}} \eta^{-\frac{4}{3}} \right) \leq \eta^2, \tag{8.48}$$

if $\varepsilon > 0$ is sufficiently small, and $\eta > \varepsilon$. Letting $\eta = \varepsilon^{\frac{1}{2}}$, we obtain $J(2) \subset J(1)$. Finally, $J(1)$ is a closed, open nonempty subset of \mathbb{R} . Hence $J(1) = \mathbb{R}$, and this finishes the proof. \square

It follows from Hölder's inequality that in the situation of Proposition 8.1, one also has the estimates (8.17) with $\eta = \varepsilon$.

9. THE SOLITON CASE

In this section, we deal with the first scenario in Theorem 5.1, namely the soliton case. We prove that the soliton scenario is inconsistent with the frequency-localized Morawetz interaction estimates developed in Section 7 and compactness up to rescaling.

Proposition 9.1. *Let $u \in C(\mathbb{R}, \dot{H}^2)$ be a solution of (1.1) such that $K = \{u(t) : t \in \mathbb{R}\}$ is precompact in \dot{H}^2 up to translation. If $n = 8$, then $u = 0$. In particular the soliton scenario in Theorem 5.1 does not hold true.*

Proof. Let $u \in C(\mathbb{R}, \dot{H}^2)$ be a solution of (1.1) of energy $E(u) > 0$ such that $K = \{g_{(1,y(t))}u(t) : t \in \mathbb{R}\}$ is precompact in \dot{H}^2 . In particular we can apply Proposition 8.1 with $\varepsilon > 0$ and deduce that

$$\| |\nabla|^{-\frac{3}{4}} P_{\geq 1} u \|_{L^4(\mathbb{R}, L^4)} \lesssim 1. \quad (9.1)$$

Independently, by (8.1), we know that, for all t ,

$$\| P_{\geq 1} u(t) \|_{\dot{H}^2}^2 \gtrsim_{E(u)} E(u) - \varepsilon^2 > 0, \quad (9.2)$$

if ε is sufficiently small. Then (9.2) implies that for all v in the \dot{H}^2 -closure of K , $P_{\geq 1} v \neq 0$. Since K is precompact in \dot{H}^2 and the mapping $v \mapsto \| |\nabla|^{-\frac{3}{4}} P_{\geq 1} v \|_{L^4}$ is continuous on \dot{H}^2 , we get that there exists $\kappa > 0$ such that

$$\forall v \in K, \quad \| |\nabla|^{-\frac{3}{4}} P_{\geq 1} v \|_{L^4} \geq \kappa. \quad (9.3)$$

Now, (9.1) and (9.3) imply that

$$\kappa^4 t \lesssim \| |\nabla|^{-\frac{3}{4}} u_h \|_{L^4([0,t], L^4)}^4 \lesssim 1. \quad (9.4)$$

Letting $t \rightarrow +\infty$, we get a contradiction in (9.4). This finishes the proof of Proposition 9.1. \square

10. THE LOW-TO-HIGH CASCADE

Now, we are ready to deal with the last scenario, and to exclude the case of a low-to-high cascade solution. In order to do so, we use the estimates coming from the frequency-localized interaction Morawetz estimates developed in Section 7 to control the action of the high-frequency part of u . Then the low-frequency part obeys an analogue of (1.1) with initial data arbitrarily small. Hence one can make its \dot{S}^2 -norm small, depending on the frequency, so as to prove that it is in fact small in L^2 . Then the solution is an H^2 solution, and conservation of mass gives a contradiction. More precisely, we prove the following proposition.

Proposition 10.1. *Let $u \in C(I, \dot{H}^2)$ be a maximal lifespan solution of (1.1) such that $K = \{g_{(h(t), x(t))} u(t) : t \in I\}$ is precompact in \dot{H}^2 for some functions h, x such that $h(t) \leq h(0) = 1$, and*

$$\liminf_{t \rightarrow \sup I} h(t) = 0, \quad (10.1)$$

then if $n = 8$, we have that $u = 0$. In particular, the low-to-high cascade scenario does not hold true.

Proof. Let u be as above. Applying Proposition 6.1, we see that $I = \mathbb{R}$, and since $h \leq 1$, given $\varepsilon > 0$, we can apply Proposition 8.1 to get that (8.1) holds true. We may also suppose that (8.5) holds true. As a first step in the proof, we claim that if $\varepsilon > 0$ is sufficiently small, the following holds true for all dyadic number $M \leq 1$:

$$\|P_{\leq M}u\|_{\dot{S}^2} \lesssim M^3. \quad (10.2)$$

Fix M_0 , a dyadic number, let $m = M_0^{10}$ and let $\kappa > 0$ to be chosen later. Since we know that (10.1) holds true and that K is precompact, using (8.2) we get that there exists $t_0 > 0$ such that

$$\|P_{\leq 1}u(t_0)\|_{\dot{H}^2} \leq \kappa^2 m. \quad (10.3)$$

We claim that for any $C > 0$, if κ is sufficiently small, independently of m , then we have that, for all dyadic numbers $M \in [m, 1]$,

$$\|P_{\leq M}u\|_{\dot{S}^2(J)} \leq \kappa C (m + M^3) \quad (10.4)$$

when J is small and $t_0 \in J$. Indeed, using the Bernstein's properties (2.5), we get that, in J ,

$$\begin{aligned} \|P_{\leq M}u\|_{\dot{S}^2}^2 &\lesssim \sum_{N \leq M} N^4 \|P_N u\|_{L^\infty L^2}^2 + \sum_{N \leq M} N^6 \|P_N u\|_{L^2 L^{\frac{8}{3}}}^2 \\ &\lesssim \sum_{N \leq M} N^4 \|P_N u(t_0)\|_{L^2}^2 + |J|^2 \sum_{N \leq M} N^4 \|\partial_t P_N\|_{L^\infty L^2}^2 \\ &\quad + |J| \sum_{N \leq M} N^6 \|P_N u\|_{L^\infty L^{\frac{8}{3}}}^2 \\ &\lesssim \kappa^4 m^2 + |J|^2 \sum_{N \leq M} N^4 \|P_N \Delta^4 u\|_{L^\infty L^2}^2 \\ &\quad + |J|^2 \sum_{N \leq M} N^4 \|P_N (|u|^2 u)\|_{L^\infty L^2}^2 + |J| \sum_{N \leq M} N^8 \|P_N u\|_{L^\infty L^2}^2 \\ &\lesssim_{E(u)} \kappa^4 m^2 + M^8 |J|^2 + |J|^2 \sum_{N \leq M} N^8 \| |u|^2 u \|_{L^\infty L^{\frac{4}{3}}}^2 + |J| M^4 \\ &\lesssim_{E(u)} \kappa^4 m^2 + M^8 |J|^2 + M^4 |J|, \end{aligned}$$

and if $|J| \lesssim_{E(u), C} \kappa$, then (10.4) holds true. Now, let $J(C)$ be the maximum interval containing t_0 on which (10.4) holds true for the constant $C > 0$. We prove that $J(2) \subset J(1)$ if κ and ε are chosen sufficiently small, independently of m . Indeed, let

$$u_{\text{vlow}} = P_{\leq m}u, \text{ and } u_{\text{med}} = P_{m < \cdot < 1}u.$$

In the following, all time integrals are taken on $J = J(2)$. Applying Strichartz estimates (3.3), we get that

$$\begin{aligned} &\|P_{\leq M}u\|_{\dot{S}^2} \\ &\lesssim \|P_{\leq M}u(t_0)\|_{\dot{H}^2} + \|\nabla P_{\leq M}|u_{\text{vlow}}|^2 u_{\text{vlow}}\|_{L^2 L^{\frac{8}{5}}} \\ &\quad + \|\nabla P_{\leq M}(|u|^2 u - |u_{\text{vlow}}|^2 u_{\text{vlow}})\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \kappa m + \|P_{\leq m}u\|_{\dot{S}^2}^3 + M \|\tilde{P}_{\leq M}(|u_{\text{vlow}}|^2 |u_{\text{med}}| + |u_{\text{vlow}}|^2 |u_h|)\|_{L^2 L^{\frac{8}{5}}} \\ &\quad + M \|\tilde{P}_{\leq M}|u_{\text{med}}|^3\|_{L^2 L^{\frac{8}{5}}} + M \|\tilde{P}_{\leq M}|u_h|^3\|_{L^2 L^{\frac{8}{5}}}, \end{aligned} \quad (10.5)$$

where $\tilde{P}_{\leq M}$ is the convolution operator whose kernel is

$$\tilde{k}(x) = M^8 \hat{\psi}(Mx)^2,$$

where ψ is as in (2.4). We remark that $\tilde{P}_{\leq M}$ has nonnegative kernel and satisfies estimates similar to those of $P_{\leq M}$. In particular, (2.5) holds true with $\tilde{P}_{\leq M}$ in place of $P_{\leq M}$. By assumption we have that

$$\|P_{\leq m}u\|_{\dot{S}^2}^3 \leq (4\kappa)^3 m^3. \quad (10.6)$$

Besides, using the Bernstein's properties (2.5), and the assumption on J , we get that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{vlow}^2 u_{med}|\|_{L^2 L^{\frac{8}{5}}} &\lesssim M\|u_{vlow}\|_{L^\infty L^4}^2 \|u_{med}\|_{L^2 L^8} \\ &\lesssim M(4\kappa m)^2 \left(\sum_{m < N < 1} N^{-1} \|\nabla |P_N u|\|_{L^2 L^8} \right) \\ &\lesssim M(\kappa m)^2 \left(\sum_{m < N < 1} N^{-1} \|P_{\leq 2N} u\|_{\dot{S}^2} \right) \\ &\lesssim Mm^2 \kappa^3 \left(\sum_{m < N < 1} N^{-1} (m + N^3) \right) \\ &\lesssim m^2 \kappa^3 M \end{aligned} \quad (10.7)$$

and, similarly, using the Bernstein's properties, (2.5) and (8.17), we have that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{vlow}^2 u_h|\|_{L^2 L^{\frac{8}{5}}} &\lesssim M^2 \|u_{vlow}^2 u_h\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim M^2 \|u_{vlow}\|_{L^4 L^8}^2 \|u_h\|_{L^\infty L^2} \\ &\lesssim \kappa^2 M^2 m^2 \varepsilon. \end{aligned} \quad (10.8)$$

Independently, using the Bernstein's properties (2.5) and the definition of J , we get that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{med}|^3\|_{L^2 L^{\frac{8}{5}}} &\lesssim M^3 \|u_{med}^3\|_{L^2 L^{\frac{8}{7}}} \\ &\lesssim M^3 \left(\sum_{m < N < 1} N^{-1} \|\nabla P_N u\|_{L^6 L^{\frac{24}{7}}} \right)^3 \\ &\lesssim M^3 \left(\sum_{m < N < 1} N^{-1} \|P_{\leq 2N} u\|_{\dot{S}^2} \right)^3 \\ &\lesssim M^3 \left(2\kappa \sum_{m < N \leq 1} N^{-1} m + N^2 \right)^3 \\ &\lesssim (2\kappa)^3 M^3, \end{aligned} \quad (10.9)$$

and, using again the Bernstein's properties (2.5) and (8.1), we obtain that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_h|^3\|_{L^2 L^{\frac{8}{5}}} &\lesssim M^4 \|u_h^3\|_{L^2 L^1} \\ &\lesssim M^4 \|\nabla|^{-\frac{1}{2}} u_h\|_{L^2 L^{\frac{8}{5}}} \|\nabla|^{\frac{7}{4}} u_h\|_{L^\infty L^2}^2 \\ &\lesssim M^4 \varepsilon^3. \end{aligned} \quad (10.10)$$

Finally, with (8.2)–(10.10), we get, if $\kappa = \varepsilon$ and ε is sufficiently small, that there holds that

$$\|P_{\leq M}u\|_{\dot{S}^2} \leq \kappa (m + M^3). \quad (10.11)$$

In particular, $J(2) \subset J(1)$. Consequently, $J(1)$ is a closed, open nonempty subset of \mathbb{R} . Hence $J(1) = \mathbb{R}$. Then (10.11) gives (10.2) for $M \in (M_0^{10}, 1)$. Since M_0 can be chosen arbitrarily small, we get (10.2) for all $M \leq 1$. Now, we finish the proof of Proposition 10.1. A consequence of (10.2) is that $u \in L^\infty L^2$. Indeed, by the Bernstein's properties (2.5), $P_{\geq M}u \in L^\infty L^2$ for any dyadic M , and using (10.2), we get that, when $M \leq 1$,

$$\begin{aligned} \|P_{\leq M}u\|_{L^\infty L^2} &\leq \sum_{N \leq M} \|P_N u\|_{L^\infty L^2} \\ &\lesssim \sum_{N \leq M} N^{-2} \|P_{\leq N}u\|_{\dot{S}^2} \\ &\lesssim \sum_{N \leq M} N \lesssim M. \end{aligned} \quad (10.12)$$

Now, let $M > 0$ be an arbitrarily small dyadic number. Since (10.1) holds true, and since K is precompact in \dot{H}^2 , we can find t_0 such that

$$\begin{aligned} \|P_{M < \cdot \leq M^{-1}}u(t_0)\|_{L^2} &\leq M^{-2} \|P_{M < \cdot \leq M^{-1}}u(t_0)\|_{\dot{H}^2} \\ &\leq M^{-2} \|P_{Mh(t_0) < \cdot \leq M^{-1}h(t_0)}(g(t_0)u(t_0))\|_{\dot{H}^2} \\ &\leq M. \end{aligned} \quad (10.13)$$

Using conservation of mass, the Bernstein's properties (2.5), (10.12) and (10.13), we deduce that

$$\begin{aligned} \|u(0)\|_{L^2} &= \|u(t_0)\|_{L^2} \\ &\leq \|P_{> M^{-1}}u(t_0)\|_{L^2} + \|P_{M < \cdot \leq M^{-1}}u(t_0)\|_{L^2} + \|P_{\leq M}u\|_{L^\infty L^2} \\ &\leq M^2 E(u)^{\frac{1}{2}} + 2M. \end{aligned} \quad (10.14)$$

Since M is arbitrary, we get that $u(0) = 0$. This concludes the proof of Proposition 10.1. \square

11. ANALITICITY OF THE FLOW MAP AND SCATTERING

In view of Theorem 4.1 and Corollary 5.1, we can finish the proof of the first assertions in Theorem 1.1 with Proposition 11.1 below.

Proposition 11.1. *Let $n \leq 8$. Then, for any $t > 0$, the mapping $u_0 \mapsto u(t)$, from H^2 into H^2 , is analytic.*

Proof. We follow arguments developed in Pausader and Strauss [31] for the fourth-order wave equation. We use the implicit function theorem. In case $1 \leq n \leq 3$, the global bound on the energy gives a global bound on the L^∞ -norm of u , and hence, the nonlinear term is lipschitz. In this case the problem can be solved with basic arguments. Now we treat the case $n \geq 4$. We divide $[0, t] = \cup_{j=1}^k I_j$ into subintervals $I_j = [a_j, a_{j+1}]$ such that

$$\|\nabla u\|_{L^{\frac{n+4}{2}}(I, L^{\frac{n(n+4)}{3n+4}})} \leq \delta. \quad (11.1)$$

First, if $I = I_0 = [0, a_1]$, we consider the mapping

$$\mathcal{T}_I : H^2 \times \dot{S}^0(I) \cap \dot{S}^2(I) \rightarrow H^2 \times \dot{S}^0(I) \cap \dot{S}^2(I)$$

defined by

$$\mathcal{T}(u_0, v) = \left(u_0, t \mapsto e^{it\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} |v|^2 v(s) ds \right).$$

The map \mathcal{T} is well defined thanks to the Strichartz estimates (3.3). It is clearly analytic, and $u \in C(I, H^2)$ is a solution of (1.1) if and only if $\mathcal{T}(u(0), u) = (u(0), u)$. An application of Strichartz estimates gives that, if δ in (11.1) is sufficiently small, then

$$\|D_2 \mathcal{T}(u(0), u)\|_{\dot{S}^0 \cap \dot{S}^2 \rightarrow \dot{S}^0 \cap \dot{S}^2} < 1,$$

where D_2 denotes derivation with respect to the second argument. Consequently, $D_2(I - \mathcal{T})(u(0), u)$ is invertible, and the implicit function theorem ensures that $u_0 \mapsto u|_I$ is analytic. In particular, $u_0 \mapsto u(a_1)$, from H^2 into H^2 , is analytic. By finite induction, we get that $u_0 \mapsto u(t)$ is analytic. \square

Now, we turn to the proof of the scattering assertion of Theorem 1.1. The statement is an easy consequence of Propositions 11.2 and 11.3 below.

Proposition 11.2. *Let $5 \leq n \leq 8$. For any $u^+ \in H^2$, respectively $u^- \in H^2$, there exists a unique $u \in C(\mathbb{R}, H^2)$, solution of (1.1) such that*

$$\|u(t) - e^{it\Delta^2} u^\pm\|_{H^2} \rightarrow 0 \quad (11.2)$$

as $t \rightarrow \pm\infty$. Besides, we have that

$$\begin{aligned} M(u(0)) &= M(u^\pm), \text{ and} \\ 2E(u(0)) &= \|u^\pm\|_{H^2}^2. \end{aligned} \quad (11.3)$$

This defines two mappings $\mathcal{W}_\pm : u^\pm \mapsto u(0)$ from H^2 into H^2 , and \mathcal{W}_+ and \mathcal{W}_- are continuous in H^2 .

Proof. By time reversal symmetry, we need only to prove Proposition 11.2 for u^+ . Let $\omega(t) = e^{it\Delta^2} u^+$. Then by the Strichartz estimates (3.3), $\omega \in \dot{S}^0(\mathbb{R}) \cap \dot{S}^2(\mathbb{R})$ and, given $\delta > 0$, there exists T_δ such that, on $I = [T_\delta, +\infty)$, (11.1) holds true with ω instead of u . For $u \in \dot{S}^0(I) \cap \dot{S}^2(I)$, we define

$$\Phi(u)(t) = \omega(t) - i \int_t^\infty e^{i(t-s)\Delta^2} |u(s)|^2 u(s) ds. \quad (11.4)$$

For δ sufficiently small, Φ defines a contraction mapping on the set

$$X_{T_\delta} = \{u \in \dot{S}^0(I) \cap \dot{S}^2(I); \|\nabla u\|_{L^{\frac{n+4}{2}}(I, L^{\frac{n(n+4)}{3n+4}})} \leq 2\delta,$$

$$\|u\|_{\dot{S}^0(I)} + \|u\|_{\dot{S}^2(I)} \lesssim \|u^+\|_{H^2}\},$$

equipped with the $\dot{S}^0(I)$ -norm. Thus Φ admits a unique fixed point u . We observe that

$$u(T_\delta + t) = e^{it\Delta^2} u(T_\delta) + i \int_{T_\delta}^{T_\delta+t} e^{i(t-s)\Delta^2} |u(s)|^2 u(s) ds$$

in H^2 . Consequently, u solves (1.1) on $I = [T_\delta, +\infty)$. Hence, using the first part of Theorem 1.1, u can be extended for all times $t \in \mathbb{R}$. Now, (11.2) follows from (11.4) and the boundedness of u in \dot{S}^2 and \dot{S}^0 -norms. Uniqueness follows from

the fact that any solution of (1.1) has a restriction in X_T for some $T \geq T_\delta$, and uniqueness of the fixed point of Φ in such spaces. The continuity statements are easy adaptations of the proof of local well-posedness, see Pausader [28]. The first equality in (11.3) follows from conservation of Mass and convergence in L^2 . For the second, we remark that since $\omega \in \dot{S}^0(\mathbb{R})$ there exists a sequence of times $t_k \rightarrow +\infty$ such that $\|\omega(t_k)\|_{L^4} \rightarrow 0$. Then, using conservation of energy, we compute

$$\begin{aligned} 2E(u(0)) &= 2E(u(t_k)) \\ &= 2E(\omega(t_k)) + o(1) \\ &= \|\omega(t_k)\|_{H^2}^2 + o(1) = \|u^+\|_{H^2}^2 + o(1), \end{aligned}$$

and letting $k \rightarrow +\infty$ we get that the second equation in (11.3) holds true. This finishes the proof of Proposition 11.2. \square

Proposition 11.3. *Let $5 \leq n \leq 8$. Given any solution $u \in C(\mathbb{R}, H^2)$ of (1.1), there exist $u^\pm \in H^2$ such that (11.2) holds true. In particular \mathcal{W}_\pm are homeomorphisms of H^2 .*

Proof. In case $5 \leq n \leq 7$, the equation is subcritical, and standard developments using the decay properties of the linear propagator, conservation of mass and the usual Morawetz estimates, give that for any solution $u \in C(\mathbb{R}, H^2)$ of (1.1), there exists $C > 0$ such that

$$\|u\|_{L^4(\mathbb{R}, L^4)} \leq C.$$

On such an assertion we refer to Cazenave [5] or Lin and Strauss [26] for the second order case, and to Pausader [28] for the classical Morawetz estimates in the case of the fourth-order Schrödinger equation. Consequently, applying Strichartz estimates, we get that

$$\|u\|_{\dot{S}^0(\mathbb{R})} + \|u\|_{\dot{S}^2(\mathbb{R})} \lesssim_u 1. \quad (11.5)$$

In case $n = 8$, as a consequence of Corollary 5.1, we get that any nonlinear solution u satisfies

$$\|u\|_{Z(\mathbb{R})} \lesssim_{E(u)} 1.$$

Using the work in Pausader [28, Proposition 2.6], we then get that (11.5) holds true also when $n = 8$. Since $e^{it\Delta^2}$ is an isometry on H^2 , (11.2) is equivalent to proving that there exists $u^+ \in H^2$ such that

$$\|e^{-it\Delta^2}u(t) - u^+\|_{H^2} \rightarrow 0 \quad (11.6)$$

as $t \rightarrow +\infty$. Now we prove that $e^{-it\Delta^2}u(t)$ satisfies a Cauchy criterion. We note that Duhamel's formula gives that

$$e^{-it_1\Delta^2}u(t_1) - e^{-it_0\Delta^2}u(t_0) = i \int_{t_0}^{t_1} e^{-is\Delta^2}|u(s)|^2u(s)ds. \quad (11.7)$$

By duality, (3.3) gives that for any $s \in [0, 2]$, and any $h \in \dot{S}(\mathbb{R})$, we have that

$$\left\| \int_{\mathbb{R}} e^{-it\Delta^2}h(t)dt \right\|_{\dot{H}^s} \lesssim \|h\|_{\dot{S}^s(\mathbb{R})}. \quad (11.8)$$

Now, (11.5) and (11.8) give that the right hand side in (11.7) is like $o(1)$ in H^2 as $t_0, t_1 \rightarrow +\infty$. In particular, $e^{-it\Delta^2}u(t)$ satisfies a Cauchy criterion, and there exists $u^+ \in H^2$ such that (11.6) holds true. We also get that

$$u^+ = u_0 + i \int_0^\infty e^{-is\Delta^2}|u(s)|^2u(s)ds, \quad (11.9)$$

and u^+ is unique. The continuity statements are easy adaptations of the proof of local well-posedness, see Pausader [28]. Now, by uniqueness, we clearly have that $u(0) = \mathcal{W}_+(u^+)$, so that \mathcal{W}_+ is an homeomorphism. This ends the proof of Proposition 11.3. \square

Proof of the scattering in Theorem 1.1. Applying Propositions 11.2 and 11.3, we see that the scattering operator $S = \mathcal{W}_+ \circ \mathcal{W}_-^{-1}$ is an homeomorphism from H^2 into H^2 . Using (11.4) and (11.9), and adapting slightly the proof of Proposition 11.1, we easily see that S is analytic. This ends the proof of the scattering part in Theorem 1.1. \square

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